<u>Def</u>: IZVOD FRECHETA

$$\| F(x+h) - F(x) - P\cdot h \| = o(\|h\|) \quad , \|h\| \to 0$$

* Neka je:

x^* tačno rešenje , $F(x^*) = 0$

x_n približno rešenje

$h = x^* - x_n \quad \Rightarrow x^* = x_n + h$

$\| F(x+h) - F(x) - P\cdot h \| = o(\|h\|)$ def. operatora Frecheta P

$$\| F(x_n + h) - F(x_n) - P\cdot h \| = o(\|h\|)$$

$$\Rightarrow \| F(x_n + \underbrace{x^* - x_n}_{h}) - F(x_n) - \underbrace{F'(x_n)}_{P} \cdot \underbrace{(x^* - x_n)}_{h} \| = \underbrace{o(\underbrace{\|x^* - x_n\|}_{h})}_{\text{mala veličina} \to 0}$$

$$\Rightarrow F(x^*) - F(x_n) - F'(x_n)\cdot (x^* - x_n) \approx 0 \qquad (\|\heartsuit\| = 0 \Leftrightarrow \heartsuit = 0)$$

$$\Rightarrow F(x_n) + F'(x_n)\cdot (x^* - x_n) \approx \underbrace{F(x^*)}_{=0} \quad \text{(jer je } x^* \text{ tačno rešenje)}$$

ako postoji rešenje

$$\Rightarrow F(x_n) + F'(x_n)\cdot (x_{n+1} - x_n) \approx 0$$

$$\Rightarrow \boxed{x_{n+1} = x_n - [F'(x_n)]^{-1} \cdot F(x_n)} \quad , n = 0, 1, \dots$$

NEWTON - RAPHSONOVA METODA.

Dokaz Leme 1

C KONVEKSAN $\Rightarrow \forall x, y \in C, 0 \le t \le 1 : tx + (1-t)y \in C$

$$tx + y - ty = y + t(x-y)$$

DEFINIŠIMO FUNKCIJU $\phi(t) = F(y + t(x-y))$

ϕ je definisana $\forall x, y \in C, t \in [0,1]$, $\phi : [0,1] \to Y$

ϕ je diferencijabilna $\forall t \in [0,1]$: $\phi'(t) = F'(y + t(x-y))\cdot(x-y)$

$\phi(0) = F(y)$

$\phi(1) = F(x)$

$\phi'(0) = F'(y)\cdot(x-y)$

$$F(x) - F(y) - F'(y)(x-y) = \underbrace{\phi(1) - \phi(0)} - \phi'(0)$$

$$= \int_0^1 \phi'(t)\,dt - \int_0^1 \phi'(0)\,dt$$

$$= \int_0^1 \underbrace{(\phi'(t) - \phi'(0))}_{\text{posmatramo ovo}}\,dt$$

$$\|\phi'(t)-\phi'(0)\| = \|F'(y+t(x-y))(x-y) - F'(y)(x-y)\|$$
$$= \|(F'(y+t(x-y)) - F'(y))\cdot(x-y)\|$$
$$\leq \|F'(y+t(x-y)) - F'(y)\|\cdot\|x-y\|$$
$$= \|F'(y)+t\cdot F'(x) - t\cdot F'(y) - F'(y)\|\ \|x-y\|$$
$$= t\ \underbrace{\|F'(x)-F'(y)\|}_{\leq \gamma\|x-y\|}\cdot\|x-y\|$$
$$\leq \gamma t\|x-y\|^2$$

$$\Rightarrow \|F(x)-F(y)-F'(y)\cdot(x-y)\| \leq \int_0^1 \|\phi'(t)-\phi'(0)\|\,dt$$
$$\leq \int_0^1 \gamma t\|x-y\|^2\,dt$$
$$= \gamma\|x-y\|^2\cdot\int_0^1 t\,dt$$
$$= \frac{\gamma}{2}\|x-y\|^2 \qquad \blacksquare$$

DOKAZ TEOREME 2

(1) dokažimo prvo $x_1 \in S(x_0,r)$:

$$x_1 = x_0 - [F'(x_0)]^{-1}\cdot F(x_0)$$
$$\left.\begin{array}{l} \|x_1-x_0\| = \|\cancel{x_0} - [F'(x_0)]^{-1}\cdot F(x_0) - \cancel{x_0}\| \\ \overset{(c)}{\leq} \alpha \overset{0<h<1}{<} r \qquad (r = \frac{\alpha}{1-h}) \end{array}\right\} \Rightarrow x_1 \in S(x_0,r)$$

dokažimo da $x_n \in S(x_0,r)$:

$$\|x_{n+1}-x_n\| = \|\cancel{x_n} - [F'(x_n)]^{-1}\cdot F(x_n) - \cancel{x_n}\|$$
$$\overset{(b)}{\leq} \beta\cdot\|F(x_n)\|$$
$$= \beta\cdot\|F(x_n) - \underbrace{F(x_{n-1}) - F'(x_{n-1})\cdot(x_n-x_{n-1})}_{=0}\|$$

(ovo je formula Newtnove met.)

$$\leq \beta\cdot\frac{\gamma}{2}\|x_n-x_{n-1}\|^2$$
$$= \frac{h}{\alpha}\|x_n-x_{n-1}\|^2 \qquad /\cdot\frac{h}{\alpha}$$

$$\frac{h}{\alpha}\|x_{n+1}-x_n\| \leq \Big(\underbrace{\frac{h}{\alpha}\|x_n-x_{n-1}\|}_{\leq\left(\frac{h}{\alpha}\|x_{n-1}-x_{n-2}\|\right)^2}\Big)^2$$
$$\leq \Big(\frac{h}{\alpha}\underbrace{\|x_1-x_0\|}_{\leq\alpha}\Big)^{2^n} \leq h^{2^n}$$

$\Rightarrow \|x_{n+1} - x_n\| \leq \alpha \cdot h^{2^n - 1}$

$$\|x_{n+1} - x_0\| = \|x_{n+1} - x_n + x_n - x_{n-1} + \dots + x_1 - x_0\|$$
$$\leq \|x_{n+1} - x_n\| + \|x_n - x_{n-1}\| + \dots + \|x_1 - x_0\|$$
$$\leq \alpha \cdot h^{2^n-1} + \alpha \cdot h^{2^{n-1}-1} + \dots + \alpha \cdot h^{2^1-1} + \alpha h^{2^0-1}$$
$$= \alpha (1 + h + h^3 + \dots + h^{2^n-1})$$
$$< \alpha \cdot \sum_{k=0}^{\infty} h^k = \frac{\alpha}{1-h} = r \qquad \left(\text{geom. red}, \sum_{k=0}^{\infty} h^k = \frac{1}{1-h}\right)$$

$\Rightarrow x_{n+1} \in S(x_0, r)$

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(2) Iz (1) $\Rightarrow \|x_{n+1} - x_n\| \leq \alpha \cdot h^{2^n - 1}$

za $k > n$: $\|x_{k+1} - x_n\| = \|x_{k+1} - x_k + x_k - x_{k-1} + \dots + x_{n+1} - x_n\|$
$$\leq \|x_{k+1} - x_k\| + \dots + \|x_{n+1} - x_n\|$$
$$\leq \alpha \cdot h^{2^k-1} + \alpha \cdot h^{2^{k-1}-1} + \dots + \alpha \cdot h^{2^n-1}$$
$$\leq \alpha \cdot h^{2^n-1} \left(1 + h^{2^n} + (h^{2^n})^2 + \dots\right) \quad \text{red}$$
$$= \alpha \cdot h^{2^n-1} \cdot \frac{1}{1-h^{2^n}} \xrightarrow[n\to\infty]{0<h<1} 0$$

$\Rightarrow \{x_k\}$ Košijev

$\Rightarrow$ konvergira ka nekom $x^* \in \overline{S(x_0, r)}$ (može i na rubu)

$\Rightarrow \lim_{n\to\infty} x_n = x^*$

Treba još pokazati da $F(x^*) = 0$:

Iz (a): $\|F'(x_n) - F'(x_0)\| \leq \gamma \|x_n - x_0\| \leq \gamma r$

$$\|F'(x_n)\| - \|F'(x_0)\| \underset{\|a\|-\|b\| \leq \|a-b\|}{\leq} \|F'(x_n) - F'(x_0)\| \leq \gamma r$$

$\Rightarrow \|F'(x_n)\| \leq \underbrace{\gamma r + \|F'(x_0)\|}_{= C = \text{const}}$

$F(x_n) = F'(x_n) \cdot (x_{n+1} - x_n) \qquad / \|\cdot\|$

$\Rightarrow \|F(x_n)\| \leq \|F'(x_n)\| \cdot \|x_{n+1} - x_n\| \leq C \cdot \|\underbrace{x_{n+1} - x_n}_{\to 0}\|$ jer $\{x_n\}$ konvergira

$\Rightarrow \lim_{k\to\infty} \|F(x_k)\| = 0$

$\Rightarrow \lim_{k\to\infty} \|F(x_k)\| \overset{\text{zbog neprekidnosti } F}{=} \|F(\lim_{k\to\infty}(x_k)\| = \|F(x^*)\| = 0$

$\Rightarrow F(x^*) = 0$

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(3) $\|x_{k+1} - x_n\| \leq d \cdot \dfrac{h^{2^n-1}}{1-h^{2^n}}$ (iz (2))

$\lim_{k\to\infty} \|x_{k+1} - x_n\| = \|x^* - x_n\| \leq d \cdot \dfrac{h^{2^n-1}}{1-h^{2^n}}$ $\boxtimes$

PROCENA BROJA ITERACIJA ZA DOSTIZANJE TAČNOSTI ε

$\|x^* - x_n\| \leq d \cdot \dfrac{h^{2^n-1}}{1-h^{2^n}} \leq \varepsilon$

$h^{2^n} \dfrac{d}{h} \leq \varepsilon (1 - h^{2^n})$

$h^{2^n} (\dfrac{d}{h} + \varepsilon) < \varepsilon$

$h^{2^n} \leq \dfrac{\varepsilon \cdot h}{d + \varepsilon h}$ $/ \log_2$

$2^n \log_2 h \leq \log_2 \dfrac{\varepsilon h}{d + \varepsilon h} \overset{h<1}{\Longrightarrow} 2^n \geq \dfrac{\log_2 \frac{\varepsilon h}{d+\varepsilon h}}{\log_2 h}$

$\Rightarrow n \geq \log_2 \dfrac{\log_2 \frac{\varepsilon h}{d+\varepsilon h}}{\log_2 h}$