

# JAKOBIJEVA METODA

$$A \in \mathbb{R}^{n \times n}, A^T = A$$

$$U_{kl} = \begin{pmatrix} 1 & & & 0 \\ & \ddots & & \\ & & \cos \phi & -\sin \phi \\ 0 & & \sin \phi & \cos \phi \\ & & & \ddots \end{pmatrix} \begin{matrix} \leftarrow k \\ \\ \leftarrow l \end{matrix}$$

$\begin{matrix} \uparrow & \uparrow \\ k & l \end{matrix}$

realna matrica rotacije

Given:

$$A \sim C, B = A \cdot U_{kl}, C = U_{kl}^* \cdot B$$

$$(1) c_{kj} = b_{kj} \cdot d + b_{lj} \beta$$

$$(2) c_{lj} = -b_{kj} \beta + b_{lj} d$$

$$c_{ij} = b_{ij}, i \neq k, l$$

$$(3) b_{ik} = a_{ik} \cdot d + a_{il} \beta$$

$$(4) b_{il} = -a_{ik} \beta + a_{il} d$$

$$(5) b_{ij} = a_{ij}, j \neq k, l$$

$$d = \cos \phi$$

$$\beta = \sin \phi$$

Jakobi:

$$A \sim B, B = U_{kl}^T \cdot A \cdot U_{kl}$$

$$b_{kj} = a_{kj} \cdot \cos \phi + a_{lj} \cdot \sin \phi$$

$$b_{lj} = -a_{kj} \cdot \sin \phi + a_{lj} \cdot \cos \phi$$

$$b_{ik} = a_{ik} \cdot \cos \phi + a_{il} \cdot \sin \phi$$

$$b_{il} = -a_{ik} \cdot \sin \phi + a_{il} \cdot \cos \phi$$

$$b_{ij} = a_{ij}, i, j \neq k, l$$

$$a_{ij} = a_{ji} \quad (\pm \text{ bog simetern})$$

$$c_{kk} \stackrel{(1)}{=} b_{kk} \cdot d + b_{lk} \cdot \beta$$

$$\stackrel{(3)}{=} (a_{kk} d + a_{kl} \beta) \cdot d + (a_{lk} d + a_{ll} \beta) \cdot \beta$$

$$b_{kk} = a_{kk} \cdot \cos^2 \phi + 2 a_{kl} \cos \phi \sin \phi + a_{ll} \sin^2 \phi$$

$$c_{ll} \stackrel{(2)}{=} -b_{kl} \cdot \beta + b_{ll} \cdot d$$

$$\stackrel{(4)}{=} -(-a_{kk} \beta + a_{kl} d) \beta + (-a_{kl} \beta + a_{ll} d) \cdot d$$

$$b_{ll} = a_{kk} \sin^2 \phi - 2 a_{kl} \cos \phi \sin \phi + a_{ll} \cos^2 \phi$$

$$c_{kl} \stackrel{(1)}{=} b_{kl} d + b_{ll} \beta$$

$$\stackrel{(4)}{=} (-a_{kk} \beta + a_{kl} d) d + (-a_{kl} \beta + a_{ll} d) \cdot \beta$$

$$b_{kl} = -a_{kk} \sin \phi \cos \phi + a_{kl} \cos^2 \phi - a_{kl} \sin^2 \phi + a_{ll} \sin \phi \cos \phi$$

$$= a_{kl} (\cos^2 \phi - \sin^2 \phi) + (a_{ll} - a_{kk}) \sin \phi \cos \phi$$

$$= a_{kl} \cdot \cos 2\phi + \frac{1}{2} (a_{ll} - a_{kk}) \sin 2\phi$$

$$= b_{lk}$$

$$b_{kk} \rightarrow 0 \quad (b_{kk} \rightarrow 0 \text{ zbog simetrije})$$

$$b_{kk} = 0 \Rightarrow \frac{1}{2}(a_{uu} - a_{kk}) \sin 2\phi + a_{ku} \cos 2\phi = 0 \quad / : \cos 2\phi$$

$$\frac{1}{2}(a_{uu} - a_{kk}) \tan 2\phi + a_{ku} = 0$$

$$\Rightarrow \boxed{\tan(2\phi) = \frac{-a_{ku}}{\frac{1}{2}(a_{uu} - a_{kk})}} \quad \text{nestabilan} \quad (a_{uu} \approx a_{kk})$$

STABILAN ALGORITAM:

$$\boxed{\lambda = -a_{kk}}$$

$$\boxed{M = \frac{1}{2}(a_{uu} - a_{kk})}$$

$$\tan 2\phi = \frac{\lambda}{M}$$

$$\cos 2\phi = \frac{1}{\pm \sqrt{1 + \tan^2 2\phi}} = \frac{1}{\pm \sqrt{1 + \frac{\lambda^2}{M^2}}} = \frac{\pm |M|}{\sqrt{\lambda^2 + M^2}}$$

$$\sin 2\phi = \sqrt{1 - \cos^2 2\phi} = \sqrt{1 - \frac{M^2}{\lambda^2 + M^2}} = \boxed{\frac{\lambda \cdot \text{sign}(M)}{\sqrt{\lambda^2 + M^2}} = \omega}$$

$$\sin 2\phi = 2 \sin \phi \cos \phi, \quad \cos^2 \phi = \frac{1 + \cos 2\phi}{2}$$

$$\omega = 2 \sin \phi \cdot \sqrt{1 + \frac{\cos 2\phi}{2}}$$

$$\Rightarrow \sin \phi = \frac{\omega}{2 \sqrt{\frac{1}{2}(1 + \cos 2\phi)}}$$

$$+ \sin^2 2\phi + \cos^2 2\phi = 1$$

$$= \frac{\omega}{2 \sqrt{\frac{1}{2}(1 + \sqrt{1 - \sin^2 2\phi})}}$$

$$\boxed{\sin \phi = \frac{\omega}{\sqrt{2(1 + \sqrt{1 - \omega^2})}}}$$

$$\boxed{\cos \phi = \sqrt{1 - \sin^2 \phi}}$$

① Jakobijevom metodom određen niz matrica  $A_m$  konvergira ka dijagonalnoj matrici.

②

$$\begin{aligned}
 b_{kj}^2 + b_{ej}^2 &= (a_{kj} \cdot \cos \phi + a_{ej} \cdot \sin \phi)^2 + (-a_{kj} \sin \phi + a_{ej} \cdot \cos \phi)^2 \\
 &= a_{kj}^2 \cdot \cos^2 \phi + 2 a_{kj} \cdot \cos \phi \cdot a_{ej} \cdot \sin \phi + a_{ej}^2 \cdot \sin^2 \phi \\
 &\quad + a_{kj}^2 \cdot \sin^2 \phi - 2 a_{kj} \cdot a_{ej} \cdot \cos \phi \cdot \sin \phi + a_{ej}^2 \cdot \cos^2 \phi \\
 &= a_{kj}^2 (\cos^2 \phi + \sin^2 \phi) + a_{ej}^2 (\sin^2 \phi + \cos^2 \phi) \\
 &= a_{kj}^2 + a_{ej}^2 \quad (*) \\
 b_{ik}^2 + b_{ie}^2 &= a_{ik}^2 + a_{ie}^2 \\
 b_{ij} &= a_{ij} \quad ; i, j \neq k, l \quad (**)
 \end{aligned}$$

(\*), (\*\*)  $\Rightarrow$  Pri transformaciji  $A \rightsquigarrow B$  suma kvadrata vandiagonalnih elemenata iz kojih su izvezi  $a_{ki}$  i  $a_{ek}$  se ne menja.

$$\begin{pmatrix} * & (k,k) & \dots & (k,l) & * \\ & (l,k) & \dots & (l,l) & * \\ * & & & & * \end{pmatrix}$$

Kad anuliramo element na poziciji  $(l,k)$  ( $i(k,l)$  zbog simetrije):  
 $b_{kl} = b_{ek} = 0$ ; posmatramo šta se dešava na dijagonali:

$$\begin{aligned}
 b_{kk}^2 + b_{ee}^2 &= \underbrace{b_{kk}^2 + \underbrace{b_{ek}^2}_{=0}}_{(*)} + \underbrace{b_{ke}^2 + b_{ll}^2}_{(*)} = a_{kk}^2 + \underbrace{a_{ek}^2 + a_{el}^2}_{a_{el} = a_{ek}} \\
 &= a_{kk}^2 + a_{ll}^2 + 2a_{kl}^2
 \end{aligned}$$

$$\Rightarrow \left\{ \begin{aligned} \sum_{i \neq j} a_{ij}^2 &= \sum_{i \neq j} b_{ij}^2 + 2a_{kl}^2 \\ \sum_{i=j} a_{ii}^2 &= 2b_{ll}^2 - 2a_{kl}^2 \end{aligned} \right\} \begin{array}{l} \text{Norma } L_2 \text{ se ne menja,} \\ \text{suma kvadrata na dijagonali se povećava} \\ \text{za onoliko koliko se smanjuje suma} \\ \text{kvadrata van dijagonale.} \end{array}$$

$$V \equiv \sqrt{\sum_{i \neq j} a_{ij}^2}$$

Želimo da dokažemo da  $A_m \xrightarrow{m \rightarrow \infty} D$ , tj. da vandiagonalni elementi  $\rightarrow 0$ ,  
 tj.  $V_m \rightarrow 0$ .

I korak: anuliramo bar 2  $a_{ij}$ ,  $i \neq j$  za koje važi:

$$|a_{ij}| \geq \frac{V}{5} \equiv c_1, \quad i \neq j, \quad 5 - \text{data const.}$$

Za  $5 \geq n$   $\exists$  bar 1 (2 zbog simetrije) elementa koji ovo zadovoljava

$$\text{pps } \nexists |a_{ij}| < \frac{V}{5} = c_1 : \underline{V^2} = \sum_{i \neq j} a_{ij}^2 < \underbrace{n(n-1)}_{\substack{\uparrow \text{ bar vandiag. el } \\ \uparrow}} \cdot c_1^2 < n^2 \cdot c_1^2 \leq 5^2 \cdot a^2 = \underline{V^2} \quad \text{g}$$

$$\nu_1^2 = \sum_{(i,j)} b_{ij}^2 = \nu^2 - \sum_{|a_{ij}| \geq c} a_{ij}^2 \leq \nu^2 - \underbrace{2c^2}_{\exists \text{ bar } 2} = \nu^2 - 2 \frac{\nu^2}{6^2} = \left(1 - \frac{2}{6^2}\right) \nu^2 \quad (\heartsuit)$$

jer su anularali te elemente  $|a_{ij}| \geq c$

II korak: anularamo još (bar) 2 vektorska elementa sa koje po modulu važi da su  $> \frac{\nu_1}{6} \equiv c_2$

Analogno:

$$\nu_2^2 = \dots \leq \nu_1^2 - 2 \frac{\nu_1^2}{6^2} = \nu_1^2 \left(1 - \frac{2}{6^2}\right) \stackrel{(\heartsuit)}{=} \nu^2 \left(1 - \frac{2}{6^2}\right)^2$$

⋮

m-ti korak:

$$\nu_m^2 \leq \underbrace{\nu^2 \left(1 - \frac{2}{6^2}\right)}_{< 1}^m \xrightarrow{m \rightarrow \infty} 0 \quad \text{jer } 6 \geq \dim(A) \geq 2$$