$$e^{\pm ikx} = \cos kx \pm i \cdot \sin kx$$

$$\Rightarrow \sum_{j=1}^{n} e^{ikx_j} = \sum_{j=1}^{n} \cos kx_j + i \cdot \sum_{j=1}^{n} \sin kx_j$$

$$\sum_{j=1}^{n} e^{ikx_j} = \left\{ x_1 = \alpha,\ x_2 = 2\alpha, \ldots, x_n = n \cdot \alpha,\ \alpha = \frac{2\pi}{n} \Rightarrow x_j = j \cdot \alpha \right\}$$

$$= \sum_{j=1}^{n} e^{ikj\alpha} = \sum_{j=1}^{n} (e^{ik\alpha})^j = e^{ik\alpha} \cdot \underbrace{\sum_{j=0}^{n-1} (e^{ik\alpha})^j}_{\text{parcijalna suma geometrijskog reda}}$$

$$\sum_{k=0}^{n-1} a \cdot r^k = a \cdot \frac{r^n - 1}{r - 1}$$

$$= e^{ik\alpha} \cdot \frac{(e^{ik\alpha})^n - 1}{e^{ik\alpha} - 1} \overset{n\alpha = x_n = 2\pi}{=} e^{ik\alpha} \cdot \frac{e^{ik2\pi} - 1}{e^{ik\alpha} - 1} = \left\{ e^{ik2\pi} = \underbrace{\cos 2k\pi}_{1} + i \underbrace{\sin 2k\pi}_{0} \right\}$$

$$= e^{ik\alpha} \cdot \frac{1 - 1}{e^{ik\alpha} - 1} = 0$$

, kada je

$$e^{ik\alpha} - 1 \neq 0$$
$$e^{ik\alpha} \neq 1$$
$$k\alpha \neq N \cdot 2\pi, \quad N = 0, \pm 1, \pm 2 \ldots$$
$$k \cdot \frac{2\pi}{n} \neq N \cdot 2\pi$$
$$k \neq N \cdot n$$

$$\Rightarrow \sum_{j=1}^{n} \cos kx_j + i \sum_{j=1}^{n} \sin kx_j = 0$$ (i realni i imaginarni deo moraju biti 0)

$$\Rightarrow \sum_{j=1}^{n} \cos kx_j = 0 \ \wedge \ \sum_{j=1}^{n} \sin kx_j = 0, \quad k \neq n \cdot N$$

Za $k, l = 0, 1, \ldots, m, \quad n \geq 2m+1$:

$$(\cos kx, \cos lx) = \sum_{j=1}^{n} \cos kx_j \cdot \cos lx_j$$

$$= \begin{cases} \sum_j \cos 0 \cdot \cos 0 = \sum_j 1 = n, & k = l = 0 \\[2mm] \sum_j \cos^2 kx_j = \sum_j \frac{1 + \cos 2kx_j}{2} = \frac{n}{2} + \frac{1}{2} \underbrace{\sum \cos 2kx_j}_{0,\ 0 \leq k \leq m,\ 0 \leq 2k \leq 2m < n\,!} = \frac{n}{2}, & k = l \neq 0 \\[2mm] \frac{1}{2}\Big(\underbrace{\sum \cos(k+l)x_j}_{0} + \underbrace{\sum \cos(k-l)x_j}_{0}\Big) = 0, & k \neq l \end{cases}$$

$0 \leq k, l \leq m$
$0 \leq k+l \leq 2m < n$
$0 < |k-l| < n$