CHAPTER 7
Vector Spaces and Subspaces

The definition of a vector space involves an arbitrary field (see Section 6.9) whose elements are called *scalars*. We adopt the following notation (unless otherwise stated or implied):

K	the field of scalars
a, b, c, or k	the elements of K
V	the given vector space
u, v, w	the elements of V

Nothing essential is lost if the reader assumes that K is the real field **R** or the complex field **C**.

7.1 VECTOR SPACES

7.1 Define a vector space.

I Let K be a given field and let V be a nonempty set with rules of addition and scalar multiplication which assigns to any $u, v \in V$ a *sum* $U + v \in V$ and to any $u \in V$, $k \in K$ a *product* $ku \in V$. Then V is called a *vector space over* K (and the elements of V are called *vectors*) if the following axioms hold:

[A_1]: For any vectors $u, v, w \in V$, $(u + v) + w = u + (v + w)$.
[A_2]: There is a vector in V, denoted by 0 and called the *zero vector*, for which $u + 0 = u$ for any vector $u \in V$.
[A_3]: For each vector $u \in V$ there is a vector in V, denoted by $-u$, for which $u + (-u) = 0$.
[A_4]: For any vectors $u, v \in V$, $u + v = v + u$.
[M_1]: For any scalar $k \in K$ and any vectors $u, v \in V$, $k(u + v) = ku + kv$.
[M_2]: For any scalars $a, b \in K$ and any vector $u \in V$, $(a + b)u = au + bu$.
[M_3]: For any scalars $a, b \in K$ and any vector $u \in V$, $(ab)u = a(bu)$.
[M_4]: For the unit scalar $1 \in K$, $1u = u$ for any vector $u \in V$.

7.2 Show that in a vector space V. (*a*) Any sum of vectors of the form $v_1 + v_2 + \cdots + v_m$ requires no parentheses and does not depend upon the order of the summands. (*b*) The zero vector 0 is unique. (*c*) The negative $-u$ of a vector u is unique. (*d*) The cancellation law holds, that is, for any vectors $u, v, w \in V$, $u + w = v + w$ implies $u = v$.

I The first four axioms of a vector space V indicate that V is a commutative group under addition. The above properties follow from this fact.

7.3 How is subtraction defined in a vector space V?

I Subtraction is defined by $u - v \equiv u + (-v)$.

7.4 In the statement of axiom [M_2], $(a + b)u = au + bu$, which operation does each plus sign represent?

I The + in $(a + b)u$ denotes the addition of the two scalars a and b; hence it represents the addition operation in the field K. On the other hand, the + in $au + bu$ denotes the addition of the two vectors au and bu; hence it represents the operation of vector addition. Thus each + represents a different operation.

7.5 In the statement of axiom [M_3], $(ab)u = a(bu)$, which operation does each product represent?

I In $(ab)u$ the product ab of the scalars a and b denotes multiplication in the field K, whereas the product of the scalar ab and the vector u denotes scalar multiplication.

In $a(bu)$ the product bu of the scalar b and the vector u denotes scalar multiplication; also, the product of the scalar a and the vector bu denotes scalar multiplication.

7.6 Let $V = K^n$, where K is an arbitrary field. Show how V is made into a vector space over K.

Vector addition and scalar multiplication is defined by $(a_1, a_2, \ldots, a_n) + (b_1, b_2, \ldots, b_n) = (a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n)$ and $k(a_1, a_2, \ldots, a_n) = (ka_1, ka_2, \ldots, ka_n)$ where $a_i, b_i, k \in K$. The zero vector in V is the n-tuple of zeros, $0 = (0, 0, \ldots, 0)$. The proof that $V = K^n$ is a vector space is identical to the proofs in Section 1.3 for $\mathbf{R}^n$.

7.7 Let $K = \mathbf{Z}_3$, the integers modulo 3. How many elements are there in the vector space $V = K^4$?

There are three choices, 0, 1 or 2, for each of the four components of a vector in V. Hence V has $3 \cdot 3 \cdot 3 \cdot 3 = 3^4 = 81$ elements.

7.8 Let V be the set of all $m \times n$ matrices with entries from an arbitrary field K. Show how V is made into a vector space.

V is a vector space over K with respect to the operations of matrix addition and scalar multiplication. The proof of this fact is identical to the proof of Theorem 2.3 on $m \times n$ matrices over $\mathbf{R}$.

7.9 Let V be the set of all polynomials $a_0 + a_1t + a_2t^2 + \cdots + a_nt^n$ with coefficients a_i from a field K. Show how V is made into a vector space.

V is a vector space over k with respect to the usual operations of addition of polynomials and multiplication by a constant.

7.10 Show that $V = \mathbf{R}^2$ is not a vector space over $\mathbf{R}$ with respect to the following operations of vector addition and scalar multiplication: $(a, b) + (c, d) = (a + c, b + d)$ and $k(a, b) = (ka, b)$. Show that one of the axioms of a vector space does not hold.

Let $r = 1$, $s = 2$, $v = (3, 4)$. Then

$$(r + s)v = 3(3, 4) = (9, 4)$$
$$rv + sv = 1(3, 4) + 2(3, 4) = (3, 4) + (6, 4) = (9, 8)$$

Since $(r + s)v \neq rv + sv$, axiom $[M_2]$ does not hold.

7.11 Show that $V = \mathbf{R}^2$ is not a vector space over $\mathbf{R}$ with respect to the operations: $(a, b) + (c, d) = (a, b)$ and $k(a, b) = (ka, kb)$. Show that one of the axioms of a vector space does not hold.

Let $v = (1, 2)$, $w = (3, 4)$. Then

$$v + w = (1, 2) + (3, 4) = (1, 2)$$
$$w + v = (3, 4) + (1, 2) = (3, 4)$$

Since $v + w \neq w + v$, axiom $[A_4]$ does not hold.

7.12 Show that $V = \mathbf{R}^2$ is not a vector space over $\mathbf{R}$ with respect to the operations: $(a, b) + (c, d) = (a + c, b + d)$ and $k(a, b) = (k^2a, k^2b)$. Show that one of the axioms of a vector space does not hold.

Let $r = 1$, $s = 2$, $v = (3, 4)$. Then

$$(r + s)v = 3(3, 4) = (27, 36)$$
$$rv + sv = 1(3, 4) + 2(3, 4) = (3, 4) + (12, 16) = (15, 20)$$

Thus $(r + s)v \neq rv + sv$, and so axiom $[M_2]$ does not hold.

7.13 Suppose E is a field which contains a subfield K. Show how E may be viewed as a vector space over K.

Let the usual addition in E be the vector addition and let the scalar product kv of $k \in K$ and $v \in E$ be the product of k and v as elements of the field E. Then E is a vector space over K.

7.14 Is the real field $\mathbf{R}$ a vector space: (a) Over $\mathbf{Q}$? (b) Over $\mathbf{Z}$? (c) Over $\mathbf{C}$?

(a) Yes, since $\mathbf{Q}$ is a subfield of $\mathbf{R}$. (b) No, since $\mathbf{Z}$ is not a field. (c) No, since $\mathbf{C}$ is not a subfield of $\mathbf{R}$.

Mirza

Is the complex field **C** a vector space: (*a*) Over **R**? (*b*) Over **Q**? (*c*) Over **Z**? (*d*) Over **C**?

▌ (*a*) Yes, since **R** is a subfield of **C**. (*b*) Yes, since **Q** is a subfield of **C**. (*c*) No, since **Z** is not a field. (*d*) Yes, every field is a vector space over itself.

Is $\mathbf{Z}_7$ a vector space over $\mathbf{Z}_5$?

▌ No. $\mathbf{Z}_5 = \{0, 1, 2, 3, 4\}$ is not a subfield of $\mathbf{Z}_7 = \{0, 1, 2, \ldots, 6\}$ since the operations are different, e.g., $2+3=0$ in $\mathbf{Z}_5$ but $2+3 \neq 0$ in $\mathbf{Z}_7$. Hence $\mathbf{Z}_7$ is not a vector space over $\mathbf{Z}_5$.

Theorem 7.1: Let V be a vector space over a field K.
(i) For any scalar $k \in K$ and $0 \in V$, $k0 = 0$.
(ii) For $0 \in K$ and any vector $u \in V$, $0u = 0$.
(iii) If $ku = 0$, where $k \in K$ and $u \in V$, then $k = 0$ or $u = 0$.
(iv) For any $k \in K$ and any $u \in V$, $(-k)u = k(-u) = -ku$.

Prove (i) of Theorem 7.1: $k0 = 0$.

▌ By axiom $[A_2]$ with $u = 0$, we have $0 + 0 = 0$. Hence by axiom $[M_1]$, $k0 = k(0+0) = k0 + k0$. Adding $-k0$ to both sides gives the desired result.

Prove (ii) of Theorem 7.1: $0u = 0$.

▌ By a property of K, $0 + 0 = 0$. Hence by axiom $[M_2]$, $0u = (0+0)u = 0u + 0u$. Adding $-0u$ to both sides yields the required result.

Prove (iii) of Theorem 7.1: If $ku = 0$, then $k = 0$ or $u = 0$.

▌ Suppose $ku = 0$ and $k \neq 0$. Then there exists a scalar k^{-1} such that $k^{-1}k = 1$; hence $u = 1u = (k^{-1}k)u = k^{-1}(ku) = k^{-1}0 = 0$.

Prove (iv) of Theorem 7.1: $(-k)u = k(-u) = -ku$.

▌ Using $u + (-u) = 0$, we obtain $0 = k0 = k(u + (-u)) = ku + k(-u)$. Adding $-ku$ to both sides gives $-ku = k(-u)$.

Using $k + (-k) = 0$, we obtain $0 = 0u = (k + (-k))u = ku + (-k)u$. Adding $-ku$ to both sides yields $-ku = (-k)u$. Thus $(-k)u = k(-u) = -ku$.

Show that for any scalar k and any vectors u and v, $k(u - v) = ku - kv$.

▌ Use the definition of subtraction, $u - v \equiv u + (-v)$, and the result $k(-v) = -kv$ to obtain $k(u - v) = k(u + (-v)) = ku + k(-v) = ku + (-kv) = ku - kv$.

Theorem 7.2: Let K be an arbitrary field and let X be any nonempty set. Let V be the set of all functions from X into K. The sum of any two functions $f, g \in V$ is the function $f + g \in V$ defined by

$$(f+g)(x) = f(x) + g(x) \qquad \forall x \in X$$

[The symbol ∀ means "for every."] Then V is a vector space over K, that is, V satisfies the eight axioms of a vector space. [V is nonempty since X is nonempty.]

Prove V in Theorem 7.2 satisfies axiom $[A_1]$.

▌ Let $f, g, h \in V$. To show that $(f+g)+h = f+(g+h)$, it is necessary to show that the function $(f+g)+h$ and the function $f+(g+h)$ both assign the same value to each $x \in X$. Now,

$$((f+g)+h)(x) = (f+g)(x) + h(x) = (f(x) + g(x)) + h(x) \qquad \forall x \in X$$
$$(f+(g+h))(x) = f(x) + (g+h)(x) = f(x) + (g(x) + h(x)) \qquad \forall x \in X$$

But $f(x)$, $g(x)$, and $h(x)$ are scalars in the field K where addition of scalars is associative; hence $(f(x) + g(x)) + h(x) = f(x) + (g(x) + h(x))$. Accordingly, $(f+g)+h = f+(g+h)$.

Prove V in Theorem 7.2 satisfies axiom $[A_2]$.

I Let 0 denote the zero function: $0(x)=0, \ \forall x \in X$. Then for any function $f \in V$,

$$(f+0)(x)=f(x)+0(x)=f(x)+0=f(x) \qquad \forall x \in X$$

Thus $f+0=f$, and 0 is the zero vector in V.

Prove V in Theorem 7.2 satisfies axiom $[A_3]$.

I For any function $f \in V$, let $-f$ be the function defined by $(-f)(x)=f(x)$. Then,

$$(f+(f))(x)=f(x)+(-f)(x)=f(x)-f(x)=0=0(x) \qquad \forall x \in X$$

Hence $f+(-f)=0$.

Prove V in Theorem 7.2 satisfies axiom $[A_4]$.

I Let $f, g \in V$. Then

$$(f+g)(x)=f(x)+g(x)=g(x)+f(x)=(g+f)(x) \qquad \forall x \in X$$

Hence $f+g=g+f$. [Note that $f(x)+g(x)=g(x)+f(x)$ follows from the fact that $f(x)$ and $g(x)$ are scalars in the field K where addition is commutative.]

5 Prove V in Theorem 7.2 satisfies axiom $[M_1]$.

I Let $f, g \in V$ and $k \in K$. Then

$$\begin{aligned}(k(f+g))(x)&=k((f+g)(x))=k(f(x)+g(x))=kf(x)+kg(x)\\&=(kf)(x)+(kg)(x)=(kf+kg)(x) \qquad \forall x \in X\end{aligned}$$

Hence $k(f+g)=kf+kg$. [Note that $k(f(x)+g(x))=kf(x)+kg(x)$ follows from the fact that $k, f(x)$, and $g(x)$ are scalars in the field K where multiplication is distributive over addition.]

7 Prove V in Theorem 7.2 satisfies axiom $[M_2]$.

I Let $f \in V$ and $a, b \in K$. Then

$$\begin{aligned}((a+b)f)(x)&=(a+b)f(x)=af(x)+bf(x)=(af)(x)+bf(x)\\&=(af+bf)(x) \qquad \forall x \in X\end{aligned}$$

Hence $(a+b)f=af+bf$.

8 Prove V in Theorem 7.2 satisfies axiom $[M_3]$.

I Let $f \in V$ and $a, b \in K$. Then,

$$((ab)f)(x)=(ab)f(x)=a(bf(x))=a(bf)(x)=(a(bf))(x) \qquad \forall x \in X$$

Hence $(ab)f=a(bf)$.

9 Prove V in Theorem 7.2 satisfies axiom $[M_4]$.

I Let $f \in V$. Then, for the unit $1 \in K$, $(1f)(x)=1f(x)=f(x)$, $\forall x \in X$. Hence $1f=f$.

0 Let V be the set of infinite sequences $(a_1, a_2, \ldots)$ with entries from a field K. Show how V is made into a vector space.

I Vector addition in V and scalar multiplication on V is defined by

$$\begin{aligned}(a_1, a_2, \ldots)+(b_1, b_2, \ldots)&=(a_1+b_1, a_2+b_2, \ldots)\\k(a_1, a_2, \ldots)&=(ka_1, ka_2, \ldots)\end{aligned}$$

where $a_i, b_j, k \in K$. The proof that V is a vector space is similar to the proofs in Section 1.3 for $\mathbf{R}^n$.

7.31 What is the zero vector 0 and the negative of a vector $u = (a_1, a_2, \ldots)$ in the vector space V of Problem 7.30?

$0 = (0, 0, \ldots)$, the sequences of 0s, and $-u = (-a_1, a_2, \ldots)$, the sequence of negatives of the entries in u.

7.32 Let V be the set of ordered pairs (a, b) of real numbers with addition in V and scalar multiplication on V defined by $(a, b) + (c, d) = (a + c, b + d)$ and $k(a, b) = (ka, 0)$. Which of the eight axioms of a vector space are satisfied by V?

V satisfies all of the axioms of a vector space except $[M_4]$: $1u = u$.

7.33 Show that axiom $[M_4]$ is not a consequence of the other axioms of a vector space.

Since the algebraic structure V in Problem 7.32 satisfies all the axioms except $[M_4]$, one cannot derive $[M_4]$ from the other axioms.

7.34 Suppose E is a field containing a subfield K. Show how the set $V = E^n$ may be viewed as a vector space over K.

Define vector addition and scalar multiplication in V as follows:

$$(a_1, a_2, \ldots, a_n) + (b_1, b_2, \ldots, b_n) = (a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n)$$
$$k(a_1, a_2, \ldots, a_n) = (ka_1, ka_2, \ldots, ka_n)$$

where $a_i, b_j \in E$ and $k \in K$. Then V is a vector space over K. [This vector space is different than the vector space E^n viewed as a vector space over E.]

7.35 Can $\mathbf{C}^2$ (pairs of complex numbers) be defined as a vector space: (*a*) Over $\mathbf{R}$? (*b*) Over $\mathbf{Q}$? (*c*) Over $\mathbf{C}$? (*d*) Over $\mathbf{Z}$?

By Problem 7.34: (*a*) yes, (*b*) yes, (*c*) yes. (*d*) Since $\mathbf{Z}$ is not a field, no.

7.36 Can $\mathbf{R}^2$ be defined as a vector space: (*a*) Over $\mathbf{Q}$? (*b*) Over $\mathbf{R}$? (*c*) Over $\mathbf{C}$?

By Problem 7.34: (*a*) yes, (*b*) yes. (*c*) Since $\mathbf{C}$ is not a subfield of $\mathbf{R}$, no.

7.37 How are "dot product," length, and orthogonality defined in an abstract vector space V?

The dot product, and related notions of length and orthogonality, are not considered as part of the fundamental vector space structure, but as an additional structure which may or may not be introduced. Such spaces shall be investigated in Chapters 14 and 20.

7.2 SUBSPACES OF VECTOR SPACES

7.38 Define a subspace of a vector space.

Let W be a subset of a vector space over a field K. W is called a *subspace* of V if W is itself a vector space over K with respect to the operations of vector addition and scalar multiplication V.

Theorem 7.3: W is a subspace of V if and only if
(i) W is nonempty (or: $0 \in W$).
(ii) W is closed under vector addition: $v, w \in W$ implies $v + w \in W$.
(iii) W is closed under scalar multiplication: $v \in W$ implies $kv \in W$ for every $k \in K$.

7.39 Prove Theorem 7.3.

Suppose W satisfies (i), (ii), and (iii). By (i), W is nonempty; and by (ii) and (iii), the operations of vector addition and scalar multiplication are well-defined for W. Moreover, the axioms $[A_1]$, $[A_4]$, $[M_1]$, $[M_2]$, $[M_3]$, and $[M_4]$ hold in W since the vectors in W belong to V. Hence we need only show that $[A_2]$ and $[A_3]$ also hold in W. By (i), W is nonempty, say $u \in W$. Then by (iii), $0u = 0 \in W$ and $v + 0 = v$ for every $v \in W$. Hence W satisfies $[A_2]$. Lastly, if $v \in W$ then $(-1)v = -v \in W$ and $v + (-v) = 0$; hence W satisfies $[A_3]$. Thus W is a subspace of V.

Conversely, if W is a subspace of V then clearly (i), (ii), and (iii) hold.

Corollary 7.4: W is a subspace of V if and only if (i) $0 \in W$ (or $W \neq \emptyset$) and (ii) $v, w \in W$ implies $av + bw \in W$ for every $a, b \in K$.

Prove Corollary 7.4.

I Suppose W satisfies (i) and (ii). Then, by (i), W is nonempty. Furthermore, if $v, w \in W$ then, by (ii), $v + w = 1v + 1w \in W$; and if $v \in W$ and $k \in K$ then, by (ii), $kv = kv + 0v \in W$. Thus by Theorem 7.3, W is a subspace of V.

Conversely, if W is a subspace of V, then clearly (i) and (ii) hold in W.

Let V be any vector space. Describe the "smallest" and "largest" subspaces of V.

I The set $\{0\}$ consisting of the zero vector alone is a subspace of V contained in every other subspace of V, and the entire space V is a subspace of V which contains every other subspace of V.

Problems 7.42–7.46 refer to the vector space $V = \mathbf{R}^3$.

Show that W is a subspace of $V = \mathbf{R}^3$ where W is the xy plane which consists of those vectors whose third component is 0, i.e., $W = \{(a, b, 0): a, b \in \mathbf{R}\}$.

I $0 = (0, 0, 0) \in W$ since the third component of 0 is 0. For any vectors $v = (a, b, 0)$, $w = (c, d, 0)$ in W, and any scalars (real numbers) k and k', $kv + k'w = k(a, b, 0) + k'(c, d, 0) = (ka, kb, 0) + (k'c, k'd, 0) = (ka + k'c, kb + k'd, 0)$. Thus $kv + k'w \in W$, and so W is a subspace of V.

Show that W is a subspace of $V = \mathbf{R}^3$ where W consists of those vectors each whose sum of components is zero, i.e., $W = \{(a, b, c): a + b + c = 0\}$.

I $0 = (0, 0, 0) \in W$ since $0 + 0 + 0 = 0$. Suppose $v = (a, b, c)$, $w = (a', b', c')$ belong to W, i.e., $a + b + c = 0$ and $a' + b' + c' = 0$. Then for any scalars k and k', $kv + k'w = k(a, b, c) + k'(a', b', c') = (ka, kb, kc) + (k'a', k'b', k'c') = (ka + k'a', kb + k'b', kc + k'c')$ and, furthermore,

$$(ka + k'a') + (kb + k'b') + (kc + k'c') = k(a + b + c) + k'(a' + b' + c') = k0 + k'0 = 0.$$

Thus $kv + k'w \in W$, and so W is a subspace of V.

Show that W is not a subspace of $V = \mathbf{R}^3$ where W consists of those vectors whose first component is nonnegative, i.e., $W = \{(a, b, c): a \geq 0\}$.

I Show that one of the properties of, say, Theorem 7.3 does not hold. $v = (1, 2, 3) \in W$ and $k = -5 \in \mathbf{R}$. But $kv = -5(1, 2, 3) = (-5, -10, -15)$ does not belong to W since -5 is negative. Hence W is not a subspace of V.

Show that W is not a subspace of $V = \mathbf{R}^3$ where W consists of those vectors whose length does not exceed 1, i.e., $W = \{(a, b, c): a^2 + b^2 + c^2 \leq 1\}$.

I $v = (1, 0, 0) \in W$ and $w = (0, 1, 0) \in W$. But $v + w = (1, 0, 0) + (0, 1, 0) = (1, 1, 0)$ does not belong to W since $1^2 + 1^2 + 0^2 = 2 > 1$. Hence W is not a subspace of V.

Show that W is not a subspace of $V = \mathbf{R}^3$ where W consists of those vectors whose components are rational numbers, i.e., $W = \{(a, b, c): a, b, c \in \mathbf{Q}\}$.

I $v = (1, 2, 3) \in W$ and $k = \sqrt{2} \in \mathbf{R}$. But $kv = \sqrt{2}(1, 2, 3) = (\sqrt{2}, 2\sqrt{2}, 3\sqrt{2})$ does not belong to W since its components are not rational numbers. Hence W is not a subspace of V.

Problems 7.47–7.48 refer to the vector space V of all n-square matrices over a field K.

Show that W is a subspace of V where W consists of the symmetric matrices, i.e., all matrices $A = (a_{ij})$ for which $a_{ij} = a_{ji}$.

▮ $0 \in W$ since all entries of 0 are 0 and hence equal. Now suppose $A = (a_{ij})$ and $B = (b_{ij})$ belong to W, i.e., $a_{ji} = a_{ij}$ and $b_{ji} = b_{ij}$. For any scalars $a, b \in K$, $aA + bB$ is the matrix whose ij-entry is $aa_{ij} + bb_{ij}$. But $aa_{ji} + bb_{ji} = aa_{ij} + bb_{ij}$. Thus $aA + bB$ is also symmetric, and so W is a subspace of V.

7.48 Show that W is a subspace of V where W consists of all matrices which commute with a given matrix T; that is, $W = \{A \in V: AT = TA\}$.

▮ $0 \in W$ since $0T = 0 = T0$. Now suppose $A, B \in W$; that is, $AT = TA$ and $BT = TB$. For any scalars $a, b \in K$, $(aA + bB)T = (aA)T + (bB)T = a(AT) + b(BT) = a(TA) + b(TB) = T(aA) + T(bB) = T(aA + bB)$. Thus $aA + bB$ commutes with T, i.e., belongs to W; hence W is a subspace of V.

Problems 7.48–7.50 refer to the vector space V of all 2×2 matrices over the real field **R**.

7.49 Show that W is not a subspace of V where W consists of all matrices with zero determinant.

▮ [Recall that $\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$.] The matrices $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ belong to W since $\det(A) = 0$ and $\det(B) = 0$. But $A + B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ does not belong to W since $\det(A + B) = 1$. Hence W is not a subspace of V.

7.50 Show that W is not a subspace of V where W consists of all matrices A for which $A^2 = A$.

▮ The unit matrix $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ belongs to W since

$$I^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

But $2I = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ does not belong to W since

$$(2I)^2 = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix} \neq 2I$$

Hence W is not a subspace of V.

Problems 7.51–7.57 refer to the vector space V of all functions from the real field **R** into **R**. Here **0** denotes the zero function: $\mathbf{0}(x) = 0$, for every $x \in \mathbf{R}$.

7.51 Show that W is a subspace of V where $W = \{f: f(3) = 0\}$, i.e., W consists of those functions which map 3 into 0.

▮ $\mathbf{0} \in W$ since $\mathbf{0}(3) = 0$. Suppose $f, g \in W$, i.e., $f(3) = 0$ and $g(3) = 0$. Then for any real numbers a and b, $(af + bg)(3) = af(3) + bg(3) = a0 + b0 = 0$. Hence $af + bg \in W$, and so W is a subspace of V.

7.52 Show that W is a subspace of V where $W = \{f: f(7) = f(1)\}$, i.e., W consists of those functions which assign the same value to 7 and 1.

▮ $\mathbf{0} \in W$ since $\mathbf{0}(7) = 0 = \mathbf{0}(1)$. Suppose $f, g \in W$, i.e., $f(7) = f(1)$ and $g(7) = g(1)$. Then for any real numbers a and b, $(af + bg)(7) = af(7) + bg(7) = af(1) + bg(1) = (af + bg)(1)$. Hence $af + bg \in W$, and so W is a subspace of V.

7.53 Show that W is a subspace of V where W consists of the odd functions, i.e., those functions f for which $f(-x) = -f(x)$.

▮ $\mathbf{0} \in W$ since $\mathbf{0}(-x) = 0 = -0 = -\mathbf{0}(x)$. Suppose $f, g \in W$, i.e., $f(-x) = -f(x)$ and $g(-x) = -g(x)$. Then for any real numbers a and b, $(af + bg)(-x) = af(-x) + bg(-x) = -af(x) - bg(x) = -(af(x) + bg(x)) = -(af + bg)(x)$. Hence $af + bg \in W$, and so W is a subspace of V.

7.54 Show that W is not a subspace of V where $W = \{f: f(7) = 2 + f(1)\}$.