1.175 Find a vector w normal to the plane H containing the points P_1, P_2, and P_3.

H contains the vectors u and v determined above. Hence $u \times v$ is normal to H. The array

$$\begin{pmatrix} 1 & 3 & -4 \\ 4 & 1 & -2 \end{pmatrix} \quad \text{gives} \quad w = u \times v = (-6, +4, -16+2, 1-12) = (-2, -14, 11)$$

1.176 Give an equation for the plane H of Problem 1.175.

Use the point $P_1(1, 2, 3)$ and the normal direction w to obtain

$$-2(x-1) - 14(y-2) - 11(z-3) = 0 \qquad \text{or} \qquad 2x + 14y + 11z = 63$$

1.177 Prove *Lagrange's identity*, $\|u \times v\|^2 = (u \cdot u)(v \cdot v) - (u \cdot v)^2$.

If $u = (a_1, a_2, a_3)$ and $v = (b_1, b_2, b_3)$, then

$$\|u \times v\|^2 = (a_2b_3 - a_3b_2)^2 + (a_3b_1 - a_1b_3)^2 + (a_1b_2 - a_2b_1)^2 \tag{1}$$

$$(u \cdot u)(v \cdot v) - (u \cdot v)^2 = (a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2) - (a_1b_1 + a_2b_2 + a_3b_3)^2 \tag{2}$$

Expansion of the right-hand sides of *(1)* and *(2)* establishes the identity.

1.178 Show that $\|u \cdot v\| = \|u\| \, \|v\| \sin \theta$, where θ is the angle between u and v.

By Problem 1.79, $u \cdot v = \|u\| \, \|v\| \cos \theta$. Then, by Problem 1.177,

$$\|u \times v\|^2 = \|u\|^2 \|v\|^2 - \|u\|^2 \|v\|^2 \cos^2 \theta = \|u\|^2 \|v\|^2 (1 - \cos^2 \theta) = \|u\|^2 \|v\|^2 \sin^2 \theta$$

Taking square roots gives us our result.

CHAPTER 2

Matrix Algebra

This chapter uses letters $A, B, C, \ldots$ to denote matrices and lowercase letters $a, b, c, x, y, \ldots$ to denote scalars.* Unless otherwise stated or implied, scalars will be real numbers; in other words, the matrices will be over **R**.

2.1 MATRICES

2.1 Find the rows, columns and size of the matrix $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$.

▌ The *rows* of A are the horizontal lines of numbers; there are two of them: $(1 \quad 2 \quad 3)$ and $(4 \quad 5 \quad 6)$. The *columns* of A are the vertical lines of numbers; there are three of them:

$$\begin{pmatrix} 1 \\ 4 \end{pmatrix} \qquad \begin{pmatrix} 2 \\ 5 \end{pmatrix} \qquad \begin{pmatrix} 3 \\ 6 \end{pmatrix}$$

The *size* of A is 2×3 [read: 2 by 3], the number of rows by the number of columns.

2.2 The notation $A = (a_{ij})_{m,n}$ [or simply $A = (a_{ij})$] is used to denote an $m \times n$ matrix. What is the significance of the first subscript, i, and the second subscript, j?

▌ The scalar a_{ij} is the element of A in row i and column j. Thus, the first subscript tells the row of the element and the second subscript tells the column.

2.3 Given the matrix $A = (a_{ij})$, find the location of (*a*) a_{35}, (*b*) $a_{4,12}$, (*c*) $a_{0,11}$, (*d*) $a_{13,-4}$.

▌ (*a*) a_{35} lies in the third row and fifth column. (*b*) $a_{4,12}$ appears in row 4 and column 12. [Note that here we need a comma to distinguish the subscripts.] (*c*) and (*d*) The scalars $a_{0,11}$ and $a_{13,-4}$ cannot be elements of A, since subscripts in a matrix are, by convention, positive integers.

2.4 Given matrices A and B, when is $A = B$?

▌ Two matrices are equal if and only if they have the same size and corresponding entries are equal.

2.5 Find x, y, z, w if $\begin{pmatrix} x+y & 2z+w \\ x-y & z-w \end{pmatrix} = \begin{pmatrix} 3 & 5 \\ 1 & 4 \end{pmatrix}$.

▌ Equate corresponding entries:

$$\begin{cases} x+y=3 \\ x-y=1 \\ 2z+w=5 \\ z-w=4 \end{cases}$$

The solution of the system of equations is $x=2$, $y=1$, $z=3$, $w=-1$.

2.6 Which of the following matrices, if any, are equal?

$$A = \begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix} \qquad B = \begin{pmatrix} 2 & 3 \\ 4 & 1 \end{pmatrix} \qquad C = \begin{pmatrix} 4 & 2 \\ 1 & 3 \end{pmatrix} \qquad D = \begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}$$

▌ Although all four matrices are 2×2 and contain the scalars 1, 2, 3, 4, no two of the matrices are equal *element by element*.

2.7 The $m \times n$ zero matrix, denoted by $\mathbf{0}_{m,n}$ or simply $\mathbf{0}$, is the matrix whose elements are all zero. Find x, y, z, t if

$$\begin{pmatrix} x+y & z+3 \\ y-4 & z+w \end{pmatrix} = \mathbf{0}$$

▌ Set all entries equal to zero to obtain the system

$$x+y=0 \qquad z+3=0 \qquad y-4=0 \qquad z+w=0$$

The solution of the system is $x=-4$, $y=4$, $z=-3$, $w=3$.

2.8 The *negative* of an $m \times n$ matrix $A = (a_{ij})$ is the $m \times n$ matrix $-A \equiv (-a_{ij})$. Find the negatives of

$$A = \begin{pmatrix} 1 & -3 & 4 & 7 \\ 2 & -5 & 0 & -8 \end{pmatrix} \qquad B = \begin{pmatrix} 2 & -3 \\ -6 & 1 \end{pmatrix} \qquad 0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

I Take the negative of each element:

$$-A = \begin{pmatrix} -1 & -(-3) & -4 & -7 \\ -2 & -(-5) & -0 & -(-8) \end{pmatrix} = \begin{pmatrix} -1 & 3 & -4 & -7 \\ -2 & 5 & 0 & 8 \end{pmatrix}$$

$$-B = \begin{pmatrix} -2 & 3 \\ 6 & -1 \end{pmatrix} \qquad -0 = \begin{pmatrix} -0 & -0 & -0 \\ -0 & -0 & -0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 0$$

2.9 Show that, for any matrix A, we have $-(-A) = A$.

I $-(-A) = -(-a_{ij})_{m,n} = (-(-a_{ij}))_{m,n} = (a_{ij})_{m,n} = A$

2.10 A matrix A with only one row is called a *row matrix* or a *row vector* and is frequently denoted by $A = (a_1 \quad a_2 \quad \cdots \quad a_n)$; we omit its first subscript since it must be one. Analogously, a matrix B with only one column is called a *column matrix* or a *column vector* and is frequently denoted by

$$B = \begin{pmatrix} b_1 \\ b_2 \\ \cdots \\ b_m \end{pmatrix}$$

Discuss the difference, if any, between the following objects:

$$u = (1 \quad 2 \quad 3) \qquad \text{and} \qquad v = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

I Viewed as vectors in $\mathbf{R}^3$, u and v may be considered equal. However, as matrices, they cannot be equal, for they have different sizes.

2.2 MATRIX ADDITION AND SCALAR MULTIPLICATION

2.11 If $A = (a_{ij})_{m,n}$ and $B = (b_{ij})_{m,n}$ are matrices of the same size, their *sum* is defined as $A + B \equiv (a_{ij} + b_{ij})_{m,n}$. Find the sum of

$$A = \begin{pmatrix} 1 & -2 & 3 \\ 4 & 5 & -6 \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} 3 & 0 & 2 \\ -7 & 1 & 8 \end{pmatrix}$$

I Add corresponding entries:

$$A + B = \begin{pmatrix} 1+3 & -2+0 & 3+2 \\ 4-7 & 5+1 & -6+8 \end{pmatrix} = \begin{pmatrix} 4 & -2 & 5 \\ -3 & 6 & 2 \end{pmatrix}$$

2.12 Find $A + B$ if $A = \begin{pmatrix} 1 & 2 & -3 \\ 0 & -4 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & 5 \\ 1 & -2 \end{pmatrix}$.

I The sum is not defined, since the matrices have different sizes.

2.13 Find $A + B$ for $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & -1 & 2 \\ 0 & 3 & -5 \end{pmatrix}$.

I Add corresponding elements:

$$A + B = \begin{pmatrix} 1+1 & 2+(-1) & 3+2 \\ 4+0 & 5+3 & 6+(-5) \end{pmatrix} = \begin{pmatrix} 2 & 1 & 5 \\ 4 & 8 & 1 \end{pmatrix}$$

2.14 Add $C = \begin{pmatrix} 1 & 2 & -3 & 4 \\ 0 & -5 & 1 & -1 \end{pmatrix}$ and $D = \begin{pmatrix} 3 & -5 & 6 & -1 \\ 2 & 0 & -2 & -3 \end{pmatrix}$.

I $$C + D = \begin{pmatrix} 1+3 & 2+(-5) & (-3)+6 & 4+(-1) \\ 0+2 & (-5)+0 & 1+(-2) & (-1)+(-3) \end{pmatrix} = \begin{pmatrix} 4 & -3 & 3 & 3 \\ 2 & -5 & -1 & -4 \end{pmatrix}$$

2.15 Redefine the negative of a matrix [Problem 2.8] in terms of matrix addition.

▌ The negative of a given matrix A is the [unique] matrix whose sum with A is the zero matrix, that is, $A+(-A)=0$. [Note that this way of defining $-A$ avoids reference to the elements of A.]

2.16 If $A=(a_{ij})_{m,n}$ and k is a scalar, the matrix $kA \equiv (ka_{ij})_{m,n}$ is called the *product* of A by the scalar k. Find $3A$ and $-5A$, where

$$A=\begin{pmatrix}1 & -2 & 3\\ 4 & 5 & -6\end{pmatrix}$$

▌ Multiply each entry by the given scalar:

$$3A=\begin{pmatrix}3\cdot 1 & 3\cdot(-2) & 3\cdot 3\\ 3\cdot 4 & 3\cdot 5 & 3\cdot(-6)\end{pmatrix}=\begin{pmatrix}3 & -6 & 9\\ 12 & 15 & -18\end{pmatrix}$$

$$-5A=\begin{pmatrix}-5\cdot 1 & -5\cdot(-2) & -5\cdot 3\\ -5\cdot 4 & -5\cdot 5 & -5\cdot(-6)\end{pmatrix}=\begin{pmatrix}-5 & 10 & -15\\ -20 & -25 & 30\end{pmatrix}$$

2.17 Compute: (*a*) $3\begin{pmatrix}2 & 4\\ -3 & 1\end{pmatrix}$, (*b*) $-2\begin{pmatrix}1 & 7\\ 2 & -3\\ 0 & -1\end{pmatrix}$.

▌ (*a*)

$$3\begin{pmatrix}2 & 4\\ -3 & 1\end{pmatrix}=\begin{pmatrix}3\cdot 2 & 3\cdot 4\\ 3\cdot(-3) & 3\cdot 1\end{pmatrix}=\begin{pmatrix}6 & 12\\ -9 & 3\end{pmatrix}$$

(*b*)

$$-2\begin{pmatrix}1 & 7\\ 2 & -3\\ 0 & -1\end{pmatrix}=\begin{pmatrix}(-2)\cdot 1 & (-2)\cdot 7\\ (-2)\cdot 2 & (-2)\cdot(-3)\\ (-2)\cdot 0 & (-2)\cdot(-1)\end{pmatrix}=\begin{pmatrix}-2 & -14\\ -4 & 6\\ 0 & 2\end{pmatrix}$$

2.18 The *difference*, $A-B$, of two matrices A and B of the same size is defined by $A-B\equiv A+(-B)$. Find $A-B$ if

$$A=\begin{pmatrix}4 & -5 & 6\\ 2 & 3 & -1\end{pmatrix} \quad\text{and}\quad B=\begin{pmatrix}2 & -3 & 8\\ 1 & -2 & -6\end{pmatrix}$$

▌

$$A-B=A+(-B)=\begin{pmatrix}4 & -5 & 6\\ 2 & 3 & -1\end{pmatrix}+\begin{pmatrix}-2 & 3 & -8\\ -1 & 2 & 6\end{pmatrix}=\begin{pmatrix}2 & -2 & -2\\ 1 & 5 & 5\end{pmatrix}$$

2.19 Find $2A-3B$, where $A=\begin{pmatrix}1 & -2 & 3\\ 4 & 5 & -6\end{pmatrix}$ and $B=\begin{pmatrix}3 & 0 & 2\\ -7 & 1 & 8\end{pmatrix}$.

▌ First perform the scalar multiplications, and then a matrix addition:

$$2A-3B=\begin{pmatrix}2 & -4 & 6\\ 8 & 10 & -12\end{pmatrix}+\begin{pmatrix}-9 & 0 & -6\\ 21 & -3 & -24\end{pmatrix}=\begin{pmatrix}-7 & -4 & 0\\ 29 & 7 & -36\end{pmatrix}$$

[Note that we multiply B by -3 and then add, rather than multiplying B by 3 and subtracting. This usually avoids errors.]

2.20 If $A=\begin{pmatrix}2 & -5 & 1\\ 3 & 0 & -4\end{pmatrix}$, $B=\begin{pmatrix}1 & -2 & -3\\ 0 & -1 & 5\end{pmatrix}$, $C=\begin{pmatrix}0 & 1 & -2\\ 1 & -1 & -1\end{pmatrix}$, find $3A+4B-2C$.

▌ First perform the scalar multiplications, and then the matrix additions:

$$3A+4B-2C=\begin{pmatrix}6 & -15 & 3\\ 9 & 0 & -12\end{pmatrix}+\begin{pmatrix}4 & -8 & -12\\ 0 & -4 & 20\end{pmatrix}+\begin{pmatrix}0 & -2 & 4\\ -2 & 2 & 2\end{pmatrix}=\begin{pmatrix}10 & -25 & -5\\ 7 & -2 & 10\end{pmatrix}$$

2.21 Find x, y, z, and w, if $3\begin{pmatrix}x & y\\ z & w\end{pmatrix}=\begin{pmatrix}x & 6\\ -1 & 2w\end{pmatrix}+\begin{pmatrix}4 & x+y\\ z+w & 3\end{pmatrix}$.

▮ First write each side as a single matrix:

$$\begin{pmatrix} 3x & 3y \\ 3z & 3w \end{pmatrix} = \begin{pmatrix} x+4 & x+y+6 \\ z+w-1 & 2w+3 \end{pmatrix}$$

Set corresponding entries equal to each other to obtain the system of four equations,

$$\begin{array}{lcl} 3x = x+4 & & 2x = 4 \\ 3y = x+y+6 & \text{or} & 2y = 6+x \\ 3z = z+w-1 & & 2z = w-1 \\ 3w = 2w+3 & & w = 3 \end{array}$$

The solution is: $x=2$, $y=4$, $z=1$, $w=3$.

2.22 Let $B = \begin{pmatrix} 5 & -2 \\ 4 & 7 \end{pmatrix}$ and $C = \begin{pmatrix} 1 & 2 \\ 6 & -3 \end{pmatrix}$. Find $A = \begin{pmatrix} x & y \\ z & w \end{pmatrix}$ such that $2A = 3B - 2C$.

▮ **Method 1.** First compute $3B - 2C$:

$$3B - 2C = \begin{pmatrix} 15 & -6 \\ 12 & 21 \end{pmatrix} + \begin{pmatrix} -2 & -4 \\ -12 & 6 \end{pmatrix} = \begin{pmatrix} 13 & -10 \\ 0 & 27 \end{pmatrix}$$

Then set $2A = 3B - 2C$:

$$\begin{pmatrix} 2x & 2y \\ 2z & 2w \end{pmatrix} = \begin{pmatrix} 13 & -10 \\ 0 & 27 \end{pmatrix}$$

Equate corresponding entries: $2x = 13$, $2y = -10$, $2z = 0$, $2w = 27$. Hence $x = 13/2$, $y = -5$, $z = 0$, and $w = 27/2$; that is,

$$A = \begin{pmatrix} 13/2 & -5 \\ 0 & 27/2 \end{pmatrix}$$

Method 2. Apply Theorem 2.1 [proved in Problems 2.24–2.31] to obtain directly $A = (3/2)B - C$.

2.23 Find $2A + 5B$, where $A = \begin{pmatrix} 1 & 3 \\ 2 & -5 \end{pmatrix}$ and $B = \begin{pmatrix} 4 & -3 & -6 \\ 3 & 7 & -8 \end{pmatrix}$.

▮ Although $2A$ and $5B$ are defined, the sum $2A + 5B$ is not defined since $2A$ and $5B$ have different sizes.

Theorem 2.1: Let M be the collection of all $m \times n$ matrices over a field K of scalars. Then for any matrices $A = (a_{ij})$, $B = (b_{ij})$, and $C = (c_{ij})$ in M, and any scalars k_1, k_2 in K,

(i)	$(A+B)+C = A+(B+C)$	(v)	$k_1(A+B) = k_1A + k_1B$
(ii)	$A + 0 = A$	(vi)	$(k_1+k_2)A = k_1A + k_2A$
(iii)	$A + (-A) = 0$	(vii)	$(k_1k_2)A = k_1(k_2A)$
(iv)	$A + B = B + A$	(viii)	$1A = A$

2.24 Prove (i) of Theorem 2.1

▮ The ij-entry of $A + B$ is $a_{ij} + b_{ij}$; hence, $(a_{ij} + b_{ij}) + c_{ij}$ is the ij-entry of $(A+B)+C$. The ij-entry of $B + C$ is $b_{ij} + c_{ij}$; hence, $a_{ij} + (b_{ij} + c_{ij})$ is the ij-entry of $A + (B + C)$. However, by the associative law of addition in K,

$$(a_{ij} + b_{ij}) + c_{ij} = a_{ij} + (b_{ij} + c_{ij})$$

Therefore, $(A+B)+C$ and $A+(B+C)$ have the same ij-entries, and hence $(A+B)+C = A + (B+C)$.

2.25 Prove (ii) of Theorem 2.1.

▮ The ij-entry of $A + 0$ is $a_{ij} + 0 = a_{ij}$. Therefore, $A + 0$ and A have the same ij-entries, and hence $A + 0 = A$.

2.26 Prove (iii) of Theorem 2.1.

∎ See Problem 2.15.

2.27 Prove (iv) of Theorem 2.1.

∎ The ij-entry of $A+B$ is $a_{ij}+b_{ij}$, and the ij-entry of $B+A$ is $b_{ij}+a_{ij}$. However, by the commutative law in K, $a_{ij}+b_{ij}=b_{ij}+a_{ij}$. Thus, $A+B$ and $B+A$ have the same ij-entries, and hence $A+B=B+A$.

2.28 Prove (v) of Theorem 2.1.

∎ The ij-entry of $A+B$ is $a_{ij}+b_{ij}$; hence $k_1(a_{ij}+b_{ij})$ is the ij-entry of $k_1(A+B)$. The ij-entry of k_1A is k_1a_{ij}, and the ij-entry of k_1B is k_1b_{ij}; hence $k_1a_{ij}+k_1b_{ij}$ is the ij-entry of k_1A+k_1B. However, by the distributive law in K, $k_1(a_{ij}+b_{ij})=k_1a_{ij}+k_1b_{ij}$. Therefore, $k_1(A+B)$ and k_1A and k_1B have the same ij-entries; and hence $k_1(A+B)=k_1A+k_1B$.

2.29 Prove (vi) of Theorem 2.1.

∎ As in Problem 2.28, the proof is by the distributive law in K.

2.30 Prove (vii) of Theorem 2.1.

∎ The ij-entry of $(k_1k_2)A$ is $(k_1k_2)a_{ij}$. The ij-entry of k_2A is k_2a_{ij}, and so $k_1(k_2a_{ij})$ is the ij-entry of $k_1(k_2A)$. However, by the associative law of multiplication in K, $(k_1k_2)a_{ij}=k_1(k_2a_{ij})$. Therefore, $(k_1k_2)A$ and $k_1(k_2A)$ have the same ij-entries, and hence $(k_1k_2)A=k_1(k_2A)$.

2.31 Prove (viii) of Theorem 2.1.

∎ The ij-entry of $1\cdot A$ is $1\cdot a_{ij}=a_{ij}$. Since $1\cdot A$ and A have the same ij-entries, they are equal.

2.32 Comment on the difference, if any, between the + signs in (vi) of Theorem 2.1.

∎ On the left, the + sign refers to addition of scalars in K; on the right, to addition of matrices in M

2.33 Prove that $0A=\mathbf{0}$, for any matrix A.

∎ By (viii), (vi), and (ii) of Theorem 2.1,

$$A+0A=1A+0A=(1+0)A=1A=A$$

and the proof follows upon the addition of $-A$ to both sides.

2.34 Show that $(-1)A=-A$.

∎ $A+(-1)A=1A+(-1)A=(1+(-1))A=0A=\mathbf{0}$, where the last step follows from Problem 2.33. Now add $-A$ to both sides.

2.35 Show that $A+A=2A$ and $A+A+A=3A$.

∎ Using (vi) and (viii) of Theorem 2.3, $2A=(1+1)A=1A+1A=A+A$. Similarly, $3A=(2+1)A=2A+1A=A+A+A$.

2.36 Prove that, for any positive integer n, $\sum_{k=1}^{n} A=A+A+\cdots+A=nA$.

∎ The proof is by induction on n. The case $n=1$ appears in Theorem 2.1(viii). Suppose $n>1$, and the theorem holds for $n-1$. Then

$$\sum_{k=1}^{n} A=\sum_{k=1}^{n-1} A+A=(n-1)A+1A=[(n-1)+1]A=nA$$