CHAPTER 1

INTRODUCTION

Following Einstein's formulation of the theory of special relativity (Einstein [1905]), several axiomatic systems have been proposed for Minkowski space-time. There are several reasons for developing a physical theory along axiomatic lines. One reason, which is not always made explicit, is a desire that special relativity may be better understood and more widely accepted. Another reason is that, if an axiomatic system is successful in clarifying concepts and exhibiting a small number of intuitively based assumptions, it is conceivable that some modification to one or more of the axioms might lead to an alternative theory of physical interest. Such was the case with the axiom system which Euclid [~ 300 B.C.] proposed for elementary geometry: subsequently Bolyai [1832] and Lobachevsky [1829] altered the Euclidean axiom of parallelism and discovered hyperbolic geometry! Similarly, we might expect that modification of an axiom system for Minkowski space-time could lead to a previously unknown, and possibly non-Riemannian, space-time of physical interest; a possibility which the reader is encouraged to keep in mind! (An example of a non-Minkowskian, but Riemannian, space-time which satisfies appropriately modified versions of the axioms given here, is the de-Sitter universe. The principal modification made is to Axiom I (§2.2)).

Prior to the present treatment, several axiomatic systems have already been formulated. Some authors assume the concept of a coordinate frame; in particular, Bunge [1967] has axiomatised the conventional approach due to Einstein [1905], while Suppes [1954, 1959] and Noll [1964] have based their systems on the assumption of the invariance of the quadratic form

$$\Delta x_1^2 + \Delta x_2^2 + \Delta x_3^2 - c^2\Delta t^2$$

with respect to transformations between coordinate frames. Zeeman [1964] has shown that the inhomogeneous Lorentz group is the largest group of automorphisms of Minkowski space-time. Robb [1936] formulated an axiomatic system in terms of a single relation (before-after) between the undefined elements which he called "events", however Robb's aim of mathematical simplicity is achieved by selecting his axioms on mathematical rather than physical grounds. A system of axioms has been proposed by Walker [1948, 1959], who suggested foundations for relativistic cosmology in terms of an undefined basis which involved the concepts of "particles", "light signals" and "temporal order". Walker's axiom system was not developed sufficiently to describe Minkowski space-time and was, in fact, restricted to sets of relatively stationary particles, but it succeeded in clarifying many kinematic concepts, especially those of "particle", light signal", "optical line" and "collinearity". The undefined basis of Szekeres [1968] bears some resemblance to that of Walker, although Szekeres regards both particles and light signals as objects whereas Walker regards particles as objects and light signals as particular instances of a binary "signal relation". Of these three axiomatic systems, only that of Szekeres [1968] succeeds in describing Minkowski space-time in terms of assumptions related to what one might describe as either "kinematic experience" or physical intuition.

Our intention is to describe Minkowski space-time in terms of undefined elements called "particles", a single undefined relation called the "signal relation" and eleven axioms which are intended to be in accordance with the reader's physical intuition. In a subjective sense, a particle corresponds to a freely moving observer who is capable of distinguishing between "local" events; the concept of a particle can therefore be regarded as a more basic concept than that of a "coordinate frame" (which distinguishes between different events by assigning different sets of coordinates to them): the undefined signal relation corresponds physically to "light signals". This undefined basis is similar to that of Szekeres [1968] in a physical sense and to that of Walker [1948, 1959] in a formal sense, although whereas Walker [1948, 1959] used *two* undefined relations, the "signal relation" and the "temporal order relation", the present treatment has only *one* undefined relation, the "signal relation". The notion of time ordering is implicit in the concept of the signal relation and so temporal order can be defined in terms of the signal relation. Apart from this change and certain other modifications which result in a weaker set of assumptions, the first five axioms of the present system, together with their elementary consequences, bear a strong resemblance to the excellent analysis of the concepts of "particles", "light signals" and "collinearity" given by Walker [1948]. The subsequent six axioms are essentially different from those of Walker [1948, 1959] and are believed to be original in their application to special relativity. Four of these axioms refer to sets of particles which represent "velocity space"; they resemble axioms which have been used in the study of metric geometry by authors such as Busemann [1955].

Before stating the axioms it may be as well to point out that there are several assumptions which we do not make. In particular, we do not assume the concept of a coordinate frame, we do not assume that the set of instants of each particle can be ordered by the real numbers, nor do we assume that particles and light signals move with constant speed. In the present axiomatic system, these propositions turn out to be theorems.

Three properties of Minkowski space-time are of central importance to the subsequent development. One-dimensional kinematics is in many ways analogous to plane absolute geometry, for it transpires that the concept of parallelism can be applied to particles and, furthermore, the corresponding question of uniqueness of parallelism is closely related to the uniform motion of particles. Both Robb [1936] and Szekeres [1968] observed that uniform motion implies uniqueness of parallelism but, in the present treatment, we are able to *prove* the uniqueness of parallelism and then to show that this implies the uniform motion of (freely-moving) particles, so that we need not assume Newton's first law of motion explicitly. The second important property is that, in contrast to the euclidean velocity space of Newtonian kinematics, the velocity space associated with Minkowski space-time is hyperbolic, a property which is established in the present treatment by making use of a recent characterisation of the elementary spaces by Tits [1952, 1955]. The third important property is that space-time coordinates are related to homogeneous coordinates in a three-dimensional hyperbolic space. Consequently there is an isomorphism between homogeneous Lorentz transformations and transformations of homogeneous coordinates in hyperbolic space.

Our primary aim is to clarify the foundations of special relativity so that the theory becomes as acceptable and familiar as euclidean geometry. Accordingly, the question of independence of the axioms is of secondary importance and is briefly discussed in Chapter 10. Consistency of the axioms can be easily verified by considering the usual model of Minkowski space-time. Since much of the terminology and notation is new, a listing of definitions and notation has been included before the main text.

CHAPTER 2

KINEMATIC AXIOMS FOR MINKOWSKI SPACE-TIME

§2.1 Primitive Notions

A model of Minkowski space-time will be described in terms of the following primitive notions:

(i) a set $\mathcal{P}$ whose elements are called *particles*, each particle being a set whose elements are called *instants;* and

(ii) a binary *relation* σ defined on the set of all instants.

Particles are denoted by the symbols $\underset{\sim}{Q}$, $\underset{\sim}{R}$, $\underset{\sim}{S}$, $\underset{\sim}{T}$, $\underset{\sim}{U}$, $\underset{\sim}{V}$, $\underset{\sim}{W}$, ... Instants belonging to a particle are denoted by the particle symbol together with some subscript, for example, $Q_1, Q_\alpha, Q_a, Q_x \in \underset{\sim}{Q}$. The set of all instants is denoted by $\mathcal{I}$. The binary relation σ is called the *signal relation.* An expression such as $Q_x \; \sigma \; R_y$ is to be read as *"a signal goes from* Q_x *to* R_y*"* or *"a signal leaves* Q_x *and arrives at* R_y*".*

In this axiomatic system, the undefined basis of primitive notions is an adaptation of the basis used by Walker [1948]. Walker's undefined elements are *instants* and *particles,* and the undefined relations are the *signal correspondence relation* and the *temporal order relation.* In the present treatment, the signal relation is analogous to Walker's signal correspondence relation but the temporal order relation is defined in terms of the signal relation (§2.3). That is, *the present system makes use of only one undefined binary relation,* whereas the system of Walker is expressed in terms of *two* undefined binary relations. In physical terms, particles correspond to "inertial particles" and signals correspond to "light signals".

§2.2 Existence of Signal Functions

AXIOM I (SIGNAL AXIOM)

Given particles $\underset{\sim}{Q}, \underset{\sim}{R}$ and an instant $R_y \in \underset{\sim}{R}$, there is a unique instant $Q_x \in \underset{\sim}{Q}$ such that $Q_x \ \sigma \ R_y$, and there is a unique instant $Q_z \in \underset{\sim}{Q}$ such that $R_y \ \sigma \ Q_z$.

This axiom is used in the proof of Theorems 1 (§2.4), 2 (§2.6), 3 (§2.7), 13 (§3.6), 16 (§4.1), 17 (§4.2), 32 (§6.4) and 61 (§9.5).

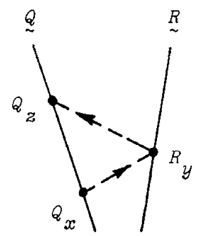

Fig. 1. In all diagrams, particles are represented by solid lines and signal relations are indicated by broken lines between instants which are represented by dots.

The Signal Axiom implies the existence of a bijection from $\underset{\sim}{Q}$ to $\underset{\sim}{R}$, which will be called a *signal function* and will be denoted by the symbol f_{RQ} where

$$f_{RQ}\colon \underset{\sim}{Q} \to \underset{\sim}{R}$$
$$Q_x \mapsto R_y \text{ if and only if } Q_x \ \sigma \ R_y.$$

Thus the Signal Axiom (Axiom I) is equivalent to the Signal Axiom of Walker [1948, Axiom S1, P322]. In the present treatment, the signal functions are equivalent to the "signal correspondences" and "signal mappings" of Walker [1948] and Walker [1959], respectively. The Axiom of Connectedness of Szekeres [1968, Axiom A4, P138] has a similar "physical content".

Given any two particles $\underset{\sim}{Q}$ and $\underset{\sim}{S}$, the composition of signal functions

$$f_{QS} \circ f_{SQ}$$

is a mapping from $\underset{\sim}{Q}$ to $\underset{\sim}{Q}$ which is related to the "motion of $\underset{\sim}{S}$ relative to $\underset{\sim}{Q}$" and is called the *record function* (of $\underset{\sim}{S}$ relative to $\underset{\sim}{Q}$).

§2.3 The Temporal Order Relation

In the present system, the "temporal (order) relation" is defined in terms of the σ-relation. This is a departure from the system of Walker [1948, P321], in which "temporal