Chapter 2

DENSITY, ELASTICITY, AND FLUIDS

IN THIS CHAPTER:

- ✔ *Simple Harmonic Motion and Springs*
- ✔ *Density and Elasticity*
- ✔ *Fluids at Rest*
- ✔ *Fluids in Motion*
- ✔ *Solved Problems*

Simple Harmonic Motion and Springs

Period

The **period** (**T**) of a cyclic system, one that is vibrating or rotating in a repetitive fashion, is the time required for the system to complete one full cycle. In the case of vibration, it is the total time for the combined back and forth motion of the system. The period is the *number of seconds per cycle.*

Frequency

The **frequency (f)** is the number of vibrations made per unit time or *the number of cycles per second*. Because T is the time for one cycle, f = 1/T. The unit of frequency is the *hertz,* where one cycle/s is one hertz (Hz).

The graph of a vibratory motion, shown in Figure 2-1, depicts up-and-down oscillation of a mass at the end of a spring. One complete cycle is from a to b, or from c to d, or from e to f. The time taken for one cycle is T, the period.

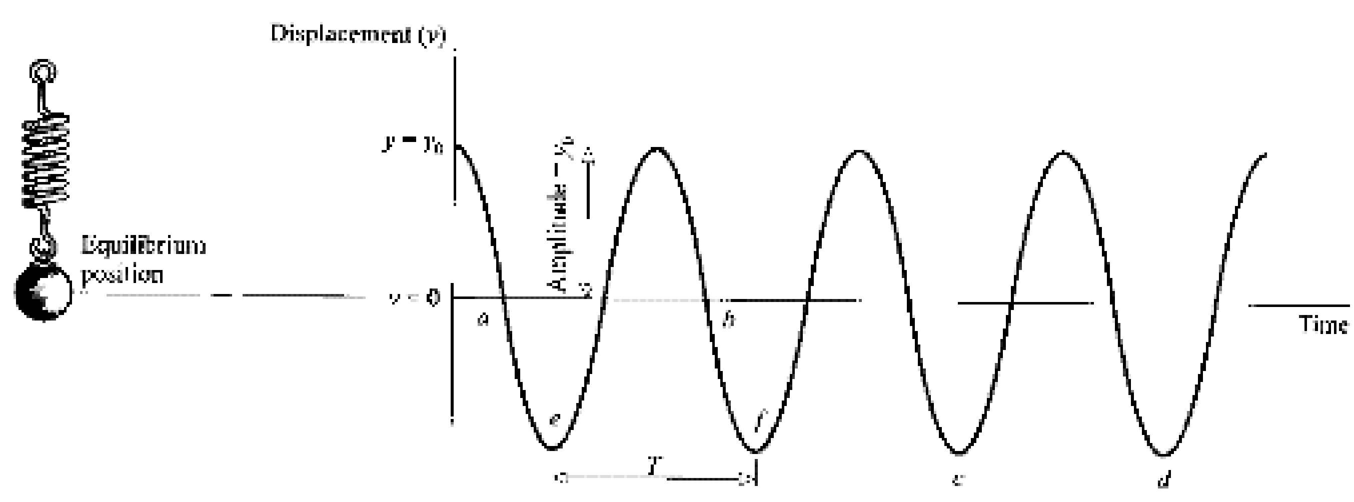


Figure 2-1

Displacement

The **displacement** (x or y) is the distance of the vibrating object from its equilibrium position (normal rest position), i.e., from the center of its vibration path. The maximum displacement is called the amplitude (see Figure 2-1).

Restoring Force

A **restoring force** is one that opposes the displacement of the system; it is necessary if vibration is to occur. In other words, a restoring force is always directed so as to push or pull the system back to its equilibrium (normal rest) position. For a mass at the end of a spring, the stretched spring pulls the mass back toward the equilibrium position, while the compressed spring pushes the mass back toward the equilibrium position.

Simple Harmonic Motion

Simple harmonic motion (**SHM**) is the vibratory motion which a system that obeys Hooke's Law undergoes. The motion illustrated in Figure 2-1 is SHM. Because of the resemblance of its graph to a sine or cosine curve, SHM is frequently called *sinusoidal motion*. A central feature of SHM is that the system oscillates at a single constant frequency. That's what makes it "simple" harmonic.

Hooke's Law

A Hookean system (a spring, wire, rod, etc.) is one that returns to its original configuration after being distorted and then released. Moreover, when such a system is stretched a distance x (for compression, x is negative), the restoring force exerted by the spring is given by **Hooke's Law**:

$F = -kx$

The minus sign indicates that the restoring force is always opposite in direction to the displacement. The **spring constant k** has units of N/m and is a measure of the stiffness of the spring. Most springs obey Hooke's Law for small distortions.

It is sometimes useful to express Hooke's Law in terms of F_{ext}, the external force needed to stretch the spring a given amount x. This force is the negative of the restoring force, and so

$F_{ext} = kx$

Elastic Potential Energy

The **elastic potential energy** stored in a Hookean spring (EPE) that is distorted a distance x is $\frac{1}{2}kx^2$. If the amplitude of motion is x_o for a mass at the end of a spring, then the energy of the vibrating system is $\frac{1}{2}kx_o^2$ at all times. However, this energy is completely stored in the spring only when $x = \pm x_0$, that is, when the mass has its maximum displacement.

Conservation of Energy

Energy interchange between kinetic and potential energy occurs constantly in a vibrating system. When the system passes through its equilibrium position, KE = maximum and EPE = 0. When the system has its maximum displacement, then KE = 0 and EPE = maximum. From the law of conservation of energy, in the absence of friction-type losses,

KE + EPE = constant

For a mass m at the end of a spring (whose own mass is negligible), this becomes

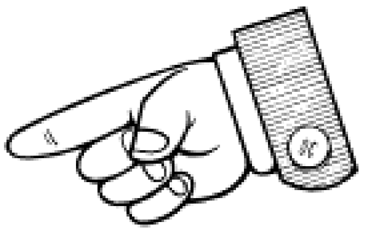
$$\frac{1}{2}mv^2 + \frac{1}{2}kx^2 = \frac{1}{2}kx_o^2$$

where x_o is the amplitude of the motion.

Motion in SHM

The speed in SHM is determined via the above energy equation as

$$|v| = \sqrt{\left(x_o^2 - x^2\right)\frac{k}{m}}$$



Acceleration in SHM is determined via Hooke's Law, F = -kx, and F = ma; once displaced and released, the restoring force drives the system. Equating these two expressions for F gives

$$a = -\frac{k}{m}x$$

The minus sign indicates that the direction of $\vec{\mathbf{a}}$ (and $\vec{\mathbf{F}}$) is always opposite to the direction of the displacement $\vec{\mathbf{x}}$. Keep in mind that neither $\vec{\mathbf{F}}$ nor $\vec{\mathbf{a}}$ is constant.

Reference Circle

Suppose that a point P moves with constant speed v_0 around a circle, as shown in Figure 2-2.

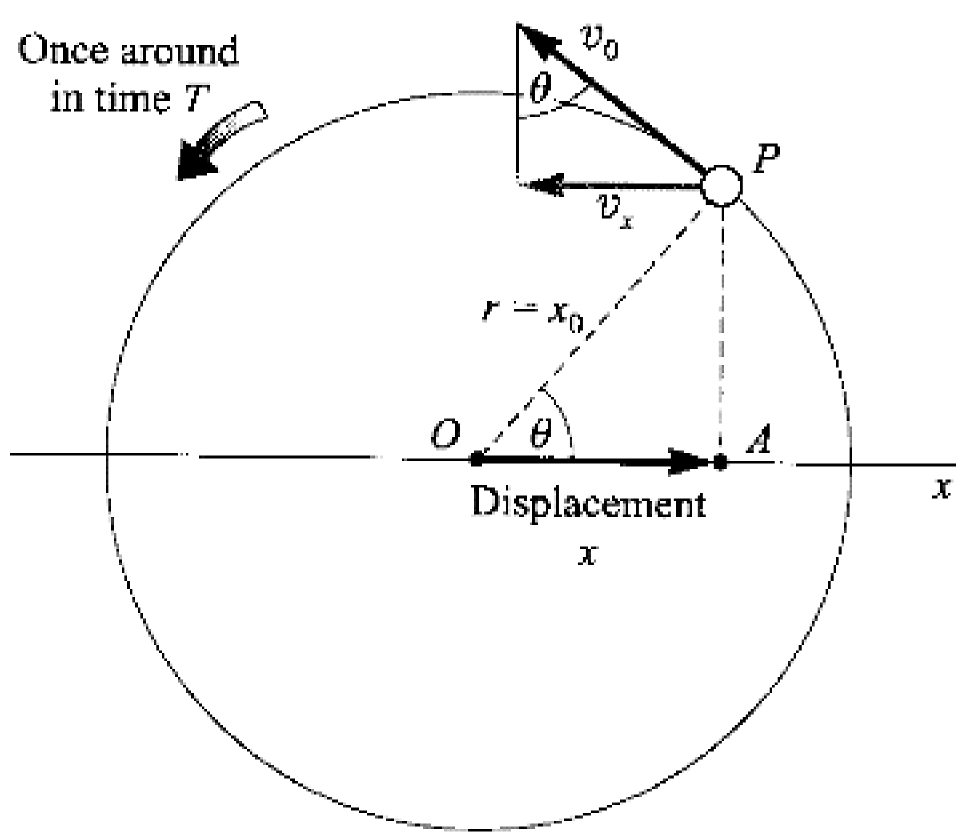


Figure 2-2

This circle is called the **reference circle** for SHM. Point A is the projection of point P on the x-axis, which coincides with the horizontal diameter of the circle. The motion of point A back and forth about point O as center is SHM. The amplitude of the motion is x_o, the radius of the circle. The time taken for P to go around the circle once is the period T of the motion. The velocity, $\vec{\mathbf{v}}_o$, of point A has a scalar component of

$$v_x = - v_o \sin \theta$$

When this quantity is positive, $\vec{\mathbf{v}}_x$, points in the positive x-direction; when it's negative, $\vec{\mathbf{v}}_x$ points in the negative x-direction.

Period in SHM

The **period T** of a SHM is the time taken for point P to go once around the reference circle in Figure 2-2. Therefore,

$$T=\frac{2\pi r}{v_o}=\frac{2\pi x_o}{v_o}$$

But v_o is the maximum speed of point A in Figure 2-2, that is, v_0 is the value of $|v_x|$ in SHM when x = 0:

$$|v_x|=\sqrt{(x_o^2-x^2)\frac{k}{m}} \quad \text{gives} \quad v_o=x_o\sqrt{\frac{k}{m}}$$

This then gives the period of SHM to be

$$T=2\pi\sqrt{\frac{m}{k}}$$

for a Hookean spring system. Eliminating the quantity k/m between the two equations a = -(k/m)x and $T=2\pi\sqrt{m/k}$, we find

$$a=-\frac{4\pi^2}{T^2}x$$

Simple Pendulum

The simple pendulum very nearly undergoes SHM if its angle of swing is not too large. The period of vibration for a pendulum of length L at a location where the gravitational acceleration is g is given by

$$T=2\pi\sqrt{\frac{L}{g}}$$

SHM can be expressed in analytic form by reference to Figure 2-2, where we see that the horizontal displacement of point P is given by $x = x_0 \cos \theta$. Since $\theta = \omega t = 2\pi ft$, where the **angular frequency** $\omega = 2\pi f$ is the angular velocity of the reference point on the circle, we have

$$x = x_0 \cos 2\pi ft = x_0 \cos \omega t$$

Similarly, the vertical component of the motion of point P is given by

$$y = x_0 \sin 2\pi ft = x_0 \sin \omega t$$

Also, from the figure, $v_x = v_0 \sin 2\pi ft$.

Density and Elasticity

Mass Density

The **mass density** (ρ) of a material is its mass per unit volume:

$$\rho = \frac{\text{mass of body}}{\text{volume of body}} = \frac{m}{V}$$

The SI unit for mass density is kg/m^3, although g/cm^3 is also used: 1000

Important Point!

Specific Gravity

The **specific gravity** (sp gr) of a substance is the ratio of the density of the substance to the density of some standard substance. The standard is usually water (at 4°C) for liquids and solids, while for gases, it is usually air.

$$\text{sp gr} = \frac{\rho}{\rho_{\text{standard}}}$$

Since sp gr is a dimensionless ratio, it has the same value for all systems of units.

$kg/m^3 = 1\ g/cm^3$. The density of water is approximately 1000 kg/m^3.

Elasticity

Elasticity is the property by which a body returns to its original size and shape when the forces that deformed it are removed.

The **stress** (σ) experienced within a solid is the magnitude of the force acting (F), divided by the area (A) over which it acts:

$$\text{Stress} = \frac{\text{force}}{\text{area of surface on which force acts}}$$

$$\sigma = \frac{F}{A}$$

Its SI unit is the pascal (Pa), where 1 Pa = 1 N/m^2. Thus, if a cane supports a load, the stress at any point within the cane is the load divided by the cross-sectional area at that point; the narrowest regions experience the greatest stress.

Strain (ε) is the fractional deformation resulting from a stress. It is measured as the ratio of the change in some dimension of a body to the original dimension in which the change occurred.

$$\text{Strain} = \frac{\text{change in dimension}}{\text{original dimension}}$$

Thus, the normal strain under an axial load is the change in length (ΔL) over the original length L_0:

$$\varepsilon = \frac{\Delta L}{L_o}$$

Strain has no units because it is a ratio of like quantities. The exact definition of strain for various situations is given later.