CHAPTER 13

Change of Basis, Similarity

13.1 CHANGE-OF-BASIS (TRANSITION) MATRIX

13.1 Define the change-of-basis matrix for a vector space V.

▌ Let $\{e_1, \ldots, e_n\}$ be a basis of V and $\{f_1, \ldots, f_n\}$ be another basis. Suppose

$$\begin{aligned} f_1 &= a_{11}e_1 + a_{12}e_2 + \cdots + a_{1n}e_n \\ f_2 &= a_{21}e_1 + a_{22}e_2 + \cdots + a_{2n}e_n \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ f_n &= a_{n1}e_1 + a_{n2}e_2 + \cdots + a_{nn}e_n \end{aligned}$$

Then the transpose P of the above matrix of coefficients is called the *change-of-basis matrix* or the *transition matrix* from the "old" basis $\{e_i\}$ to the "new" basis $\{f_i\}$. In other words, the columns of P are, respectively, the coordinates of the vectors $f_1, f_2, \ldots, f_n$ with respect to the "old" basis $\{e_i\}$.

Theorems 13.1 and 13.2, whose proofs appear in Problems 13.43 and 13.44, will be used below.

Theorem 13.1: Let P be the change-of-basis matrix from a basis $\{e_i\}$ to a basis $\{f_i\}$ and Q be the change-of-basis matrix from the basis $\{f_i\}$ back to the basis $\{e_i\}$. Then P is invertible and $Q = P^{-1}$.

Theorem 13.2: Let P be the change-of-basis matrix from a basis $\{e_i\}$ to a basis $\{f_i\}$ in a vector space V. Then, for any vector $v \in V$: (i) $P[v]_f = [v]_e$ and (ii) $P^{-1}[v_e] = [v]_f$.

Remark: Although P is called the transition matrix from the old basis $\{e_i\}$ to the new basis $\{f_i\}$, its effect is to transform the coordinates of a vector in the new basis $\{f_i\}$ back to the coordinates in the old basis $\{e_i\}$.

Problems 13.2–13.12 refer to the following bases of $\mathbf{R}^2$: $S_1 = \{u_1 = (1, -2), u_2 = (3, -4)\}$ and $S_2 = \{v_1 = (1, 3), v_2 = (3, 8)\}$. In particular, Problems 13.2–13.5 find the change-of-basis matrix P from S_1 to S_2 and Problems 13.6–13.9 find the change-of-basis matrix Q from S_2 back to S_1.

13.2 Find the coordinates of an arbitrary vector (a, b) in $\mathbf{R}^2$ with respect to the basis $S_1 = \{u_1, u_2\}$.

▌ We have

$$\begin{pmatrix} a \\ b \end{pmatrix} = x\begin{pmatrix} 1 \\ -2 \end{pmatrix} + y\begin{pmatrix} 3 \\ -4 \end{pmatrix} \quad \text{or} \quad \begin{aligned} x + 3y &= a \\ -2x - 4y &= b \end{aligned} \quad \text{or} \quad \begin{aligned} x + 3y &= a \\ 2y &= 2a + b \end{aligned}$$

Solve for x and y in terms of a and b to get $x = -2a - \frac{3}{2}b$, $y = a + \frac{1}{2}b$. Thus

$$(a, b) = (-2a - \tfrac{3}{2}b)u_1 + (a + \tfrac{1}{2}b)u_2 \quad \text{or} \quad [(a, b)]_{S_1} = [-2a - \tfrac{3}{2}b, a + \tfrac{1}{2}b]^T$$

13.3 Write v_1, the first basis vector of S_2, as a linear combination of the basis vectors u_1 and u_2 of S_1.

▌ Use Problem 13.2 to get $v_1 = (1, 3) = (-2 - \frac{9}{2})u_1 + (1 + \frac{3}{2})u_2 = (-\frac{13}{2})u_1 + (\frac{5}{2})u_2$.

13.4 Write v_2 as a linear combination of u_1 and u_2.

▌ $v_2 = (3, 8) = (-6 - 12)u_1 + (3 + 4)u_2 = -18u_1 + 7u_2$.

13.5 Find the change-of-basis matrix P from S_1 to S_2.

Write the coordinates of v_1 and v_2 in the basis S_1 as columns:

$$P = \begin{pmatrix} -\frac{13}{2} & -18 \\ \frac{5}{2} & 7 \end{pmatrix}$$

13.6 Find the coordinates of an arbitrary vector $(a, b) \in \mathbf{R}^2$ with respect to the basis $S_2 = \{v_1, v_2\}$.

We have

$$\begin{pmatrix} a \\ b \end{pmatrix} = x\begin{pmatrix} 1 \\ 3 \end{pmatrix} + y\begin{pmatrix} 3 \\ 8 \end{pmatrix} \quad \text{or} \quad \begin{aligned} x + 3y &= a \\ 3x + 8y &= b \end{aligned}$$

Solve for x and y to get $x = -8a + 3b$, $y = 3a - b$. Thus

$$(a, b) = (-8a + 3b)v_1 + (3a - b)v_2 \quad \text{or} \quad [(a, b)]S_2 = [-8a + 3b, 3a - b]^T$$

13.7 Write u_1, the first basis vector of S_1, as a linear combination of the basis vectors v_1 and v_2 of S_2.

Use Problem 13.6 to get $u_1 = (1, -2) = (-8 - 6)v_1 + (3 + 2)v_2 = -14v_1 + 5v_2$.

13.8 Write u_2 as a linear combination of v_1 and v_2.

$u_2 = (3, -4) = (-24 - 12)v_1 + (9 + 4)v_2 = -36v_1 + 13v_2$.

13.9 Find the change-of-basis matrix Q from S_2 back to S_1.

Write the coordinates of u_1 and u_2 in the basis S_2 as columns:

$$Q = \begin{pmatrix} -14 & -36 \\ 5 & 13 \end{pmatrix}$$

13.10 Verify that $Q = P^{-1}$ [Theorem 13.1].

$$QP = \begin{pmatrix} -14 & -36 \\ 5 & 13 \end{pmatrix}\begin{pmatrix} -\frac{13}{2} & -18 \\ \frac{5}{2} & 7 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

13.11 Show that $P[v]_{S_2} = [v]_{S_1}$ for any vector $v = (a, b)$ [Theorem 13.2(i)].

Using Problems 13.2, 13.5, and 13.6,

$$P[v]_{S_1} = \begin{pmatrix} -\frac{13}{2} & -18 \\ \frac{5}{2} & 7 \end{pmatrix}\begin{pmatrix} -8a + 3b \\ 3a - b \end{pmatrix} = \begin{pmatrix} -2a - \frac{3}{2}b \\ a + \frac{1}{2}b \end{pmatrix} = [v]_{S_1}$$

13.12 Show that $P^{-1}[v]_{S_1} = [v]_{S_2}$ for any vector $v = (a, b)$ [Theorem 13.2(ii)].

$$P^{-1}[v]_{S_1} = Q[v]_{S_1} = \begin{pmatrix} -14 & -36 \\ 5 & 13 \end{pmatrix}\begin{pmatrix} -2a + \frac{3}{2}b \\ a + \frac{1}{2}b \end{pmatrix} = \begin{pmatrix} -8a + 3b \\ 3a - b \end{pmatrix} = [v]_{S_2}$$

Problems 13.13–13.25 refer to the following bases of $\mathbf{R}^3$:

$$S = \{u_1 = (1, 2, 0), u_2 = (1, 3, 2), u_3 = (0, 1, 3)\} \quad \text{and} \quad S' = \{v_1 = (1, 2, 1), v_2 = (0, 1, 2), v_3 = (1, 4, 6)\}$$

In particular, Problems 13.27–13.17 find the change-of-basis matrix P from S to S', and Problems 13.18–13.22 find the change-of-basis matrix Q from B' to S.

13.13 Find the coordinates of an arbitrary vector $(a, b, c) \in \mathbf{R}^3$ with respect to the basis $S = \{u_1, u_2, u_3\}$.

We have

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = x\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + y\begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} + z\begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix} \quad \text{or} \quad \begin{aligned} x + y \quad &= a \\ 2x + 3y + z &= b \\ 2y + 3z &= c \end{aligned}$$

Solve for x, y, z to get $x = 7a - 3b + c$, $y = -6a + 3b - c$, $z = 4a - 2b + c$. Thus

$$(a, b, c) = (7a - 3b + c)u_1 + (-6a + 3b - c)u_2 + (4a - 2b + c)u_3$$

or $[(a, b, c)]_S = [7a - 3b + c,\ -6a + 3b - c,\ 4a - 2b + c]^T$.

13.14 Write v_1, the first basis vector in S', as a linear combination of the basis vectors u_1, u_2, u_3 of S.

▮ Use Problem 13.13 to get $v_1 = (1, 2, 1) = (7 - 6 + 1)u_1 + (-6 + 6 - 1)u_2 + (4 - 4 + 1)u_3 = 2u_1 - u_2 + u_3$

13.15 Write v_2 as a linear combination of u_1, u_2, and u_3.

▮ $v_2 = (0, 1, 2) = (0 - 3 + 2)u_1 + (0 + 3 - 2)u_2 + (0 - 2 + 2)u_3 = -u_1 + u_2 + 0u_3$.

13.16 Write v_3 as a linear combination of u_1, u_2, and u_3.

▮ $v_3 = (1, 4, 6) = (7 - 12 + 6)u_1 + (-6 + 12 - 6)u_2 + (4 - 8 + 6)u_3 = u_1 + 0u_2 + 2u_3$.

13.17 Find the change-of-basis matrix P from the basis S to the basis S'.

▮ Write the coordinates of v_1, v_2, and v_3 with respect to the basis S as columns:

$$P = \begin{pmatrix} 2 & -1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 2 \end{pmatrix}$$

13.18 Find the coordinates of an arbitrary vector $v = (a, b, c) \in \mathbf{R}^3$ with respect to the basis $S' = \{v_1, v_2, v_3\}$.

▮ We have

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = x\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + y\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} + z\begin{pmatrix} 1 \\ 4 \\ 6 \end{pmatrix} \quad \text{or} \quad \begin{aligned} x + \quad\ \ z &= a \\ 2x + y + 4z &= b \\ x + 2y + 6z &= c \end{aligned}$$

Solve for x, y, z to get $x = -2a + 2b - c$, $y = -8a + 5b - 2c$, $z = 3a - 2b + c$. Thus

$$v = (a, b, c) = (-2a + 2b - c)v_1 + (8a + 5b - 2c)v_2 + (3a - 2b + c)v_3$$

or $[v]_{S'} = [(a, b, c)]_{S'} = [-2a + 2b - c,\ -8a + 5b - 2c,\ 3a - 2b + c]^T$.

13.19 Write u_1, the first basis vector of S, as a linear combination of the basis vectors v_1, v_2, v_3 of S'.

▮ By Problem 13.18, $u_1 = (1, 2, 0) = (-2 + 4 + 0)v_1 + (-8 + 10 + 0)v_2 + (3 - 4 + 0)v_3 = 2v_1 + 2v_2 - v_3$.

13.20 Write u_2 as a linear combination of v_1, v_2, and v_3.

▮ $u_2 = (1, 3, 2) = (-2 + 6 - 2)v_1 + (-8 + 15 - 4)v_2 + (3 - 6 + 2)v_3 = 2v_1 + 3v_2 - v_3$.

13.21 Write u_3 as a linear combination of v_1, v_2, and v_3.

▮ $u_3 = (0, 1, 3) = (0 + 2 - 3)v_1 + (0 + 5 - 6)v_2 + (0 - 2 + 3)v_3 = -v_1 - v_2 + v_3$.

13.22 Find the change-of-basis matrix Q from the basis S' back to the basis S.

▮ Write the coordinates of u_1, u_2, and u_3 with respect to the basis S' as columns:

$$Q = \begin{pmatrix} 2 & 2 & -1 \\ 2 & 3 & -1 \\ -1 & -1 & 1 \end{pmatrix}$$

13.23 Verify that $Q = P^{-1}$ [Theorem 13.1].

$$QP = \begin{pmatrix} 2 & 2 & -1 \\ 2 & 3 & -1 \\ -1 & -1 & 1 \end{pmatrix}\begin{pmatrix} 2 & -1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I$$

13.24 Show that $P[v]_{S'} = [v]_S$ for any vector $v = (a, b, c)$ [Theorem 13.2(i)].

$$P[v]_{S'} = \begin{pmatrix} 2 & -1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 2 \end{pmatrix}\begin{pmatrix} -2a+2b-c \\ -8a+5b-2c \\ 3a-2b+c \end{pmatrix} = \begin{pmatrix} 7a+3b+c \\ -6a+3b-c \\ 4a-2b+c \end{pmatrix} = [v]_S$$

13.25 Show that $P^{-1}[v]_S = [v]_{S'}$ for any vector $v = (a, b, c)$ [Theorem 13.2(ii)].

$$P^{-1}[v]_S = Q[v]_S = \begin{pmatrix} 2 & 2 & -1 \\ 2 & 3 & -1 \\ -1 & -1 & 1 \end{pmatrix}\begin{pmatrix} 7a-3b+c \\ -6a+3b-c \\ 4a+2b+c \end{pmatrix} = \begin{pmatrix} -2a+2b-c \\ -8a+5b-2c \\ 3a-2b+c \end{pmatrix} = [v]_{S'}$$

13.26 Suppose $v_1 = (a_1, a_2, \ldots, a_n)$, $v_2 = (b_1, b_2, \ldots, b_n), \ldots$, $v_n = (c_1, c_2, \ldots, c_n)$ form a basis S of K^n. Show that the change-of-basis matrix from the usual basis $E = \{e_i\}$ of K^n to the basis S is the matrix P whose columns are the vectors $v_1, v_2, \ldots, v_n$, respectively.

We have

$$\begin{aligned} v_1 &= (a_1, a_2, \ldots, a_n) = a_1e_1 + a_2e_2 + \cdots + a_ne_n \\ v_2 &= (b_1, b_2, \ldots, b_n) = b_1e_1 + b_2e_2 + \cdots + b_ne_n \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ v_n &= (c_1, c_2, \ldots, c_n) = c_1e_1 + c_2e_2 + \cdots + c_ne_n \end{aligned}$$

Writing the coordinates as columns, we get

$$P = \begin{pmatrix} a_1 & b_1 & \cdots & c_1 \\ a_2 & b_2 & \cdots & c_2 \\ \cdots & \cdots & \cdots & \cdots \\ a_n & b_n & \cdots & c_n \end{pmatrix}$$

as claimed.

13.27 Find the change-of-basis matrix P from the usual basis $E = \{e_1, e_2, e_3\}$ of $\mathbf{R}^3$ to the basis $S = \{w_1 = (1, 1, 1), w_2 = (1, 1, 0), w_3 = (1, 0, 0)\}$.

By Problem 13.26, write the basis vectors w_1, w_2, w_3 as columns:

$$P = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

13.28 Find the change-of-basis matrix Q from the above basis S back to the usual basis E of $\mathbf{R}^3$.

Recall [Problem 12.66] that $(a, b, c) = cw_1 + (b-c)w_2 + (a-b)w_3$. Thus

$$\begin{aligned} e_1 &= (1, 0, 0) = 0w_1 + 0w_2 + 1w_3 \\ e_2 &= (0, 1, 0) = 0w_1 + 1w_2 - 1w_3 \\ e_3 &= (0, 0, 1) = 1w_1 - 1w_2 + 0w_3 \end{aligned} \qquad \text{and} \qquad Q = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \end{pmatrix}$$

13.29 Verify that $Q = P^{-1}$ for the above matrices P and Q [Theorem 13.1].

$$PQ = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I$$