Solve the system $\begin{cases} y=0 \\ x+(\pi/2)z=0 \\ -y+\pi z=0 \end{cases}$ to obtain only the zero solution: $x=0$, $y=0$, $z=0$. Hence f, g, and h are independent.

Method 2. Take the first, second, and third derivatives of $x \sin t + y \cos t + zt = 0$ with respect to t to get

$$x\cos t - y\sin t + z = 0 \qquad (1)$$
$$-x\sin t - y\cos t = 0 \qquad (2)$$
$$-x\cos t + y\sin t = 0 \qquad (3)$$

Add *(1)* and *(3)* to obtain $z=0$. Multiply *(2)* by $\sin t$ and *(3)* by $\cos t$, and then add:

$$\begin{array}{ll} \sin t \times (2): & -x\sin^2 t - y\sin t\cos t = 0 \\ \cos t \times (3): & -x\cos^2 t + y\sin t\cos t = 0 \\ \hline & -x(\sin^2 t + \cos^2 t) = 0 \quad \text{or} \quad x=0 \end{array}$$

Last, multiply *(2)* by $-\cos t$ and *(3)* by $\sin t$; and then add to obtain

$$y(\cos^2 t + \sin^2 t) = 0 \qquad \text{or} \qquad y=0$$

Since $\quad x\sin t + y\cos t + zt = 0 \quad$ implies $\quad x=0,\ y=0,\ z=0$

f, g, and h are independent.

8.27 Show that the vectors $v=(1+i, 2i)$ and $w=(1, 1+i)$ in $\mathbf{C}^2$ are linearly dependent over the complex field $\mathbf{C}$ but are linearly independent over the real field $\mathbf{R}$.

∎ Recall that two vectors are dependent iff one is a multiple of the other. Since the first coordinate of w is 1, v can be a multiple of w iff $v=(1+i)w$. But $1+i \notin \mathbf{R}$; hence v and w are independent over $\mathbf{R}$. Since $(1+i)w=(1+i)(1,1+i)=(1+i,2i)=v$ and $1+i \in \mathbf{C}$, they are dependent over $\mathbf{C}$.

8.28 Let u, v, and w be independent vectors. Show that $u+v$, $u-v$, and $u-2v+w$ are also independent.

∎ Suppose $x(u+v)+y(u-v)+z(u-2v+w)=0$ where x, y, and z are scalars. Then $xu+xv+yu-yv+zu-2zv+zw=0$ or $(x+y+z)u+(x-y-2z)v+zw=0$. But u, v, and w are linearly independent; hence the coefficients in the above relation are each 0:

$$\begin{aligned} x+y+\ z&=0 \\ x-y-2z&=0 \\ z&=0 \end{aligned}$$

The only solution to the above system is $x=0$, $y=0$, $z=0$. Hence $u+v$, $u-v$, and $u-2v+w$ are independent.

8.3 THEOREMS ON BASES AND DIMENSION

8.29 Define a basis of a vector space V.

∎ A sequence of vectors $\{u_1, u_2, \ldots, u_n\}$ is a basis of V if (1) $u_1, u_2, \ldots, u_n$ are linearly independent and (2) $u_1, u_2, \ldots, u_n$ span V.

8.30 Define the dimension of a vector space V.

∎ A vector space V is said to be of finite dimension n or to be n-dimensional, written $\dim V = n$, if V contains a basis with n elements. [This definition of dimension is well-defined by Theorem 8.4 which states that any two bases have the same number of elements.]

The vector space $\{0\}$ is defined to have dimension 0. [In a certain sense this agrees with the above definition since, by definition, $\emptyset$ is independent and generates $\{0\}$.] When a vector space is not of finite dimension, it is said to be of *infinite dimension*.

Lemma 8.1: The nonzero vectors $v_1, \ldots, v_m$ are linearly dependent if and only if one of them, say v_i, is a linear combination of the preceding vectors:

$$v_i = a_1 v_1 + \cdots + a_{i-1} v_{i-1}$$

8.31 Prove Lemma 8.1.

Suppose $v_i = a_1 v_1 + \cdots + a_{i-1} v_{i-1}$. Then $a_1 v_1 + \cdots + a_{i-1} v_{i-1} - v_i + 0v_{i+1} + \cdots + 0v_m = 0$ and the coefficient of v_i is not 0. Hence the v_i are linearly dependent.

Conversely, suppose the v_i are linearly dependent. Then there exist scalars $a_1, \ldots, a_m$, not all 0, such that $a_1 v_1 + \cdots + a_m v_m = 0$. Let k be the largest integer such that $a_k \neq 0$. Then

$$a_1 v_1 + \cdots + a_k v_k + 0v_{k+1} + \cdots + 0v_m = 0 \qquad \text{or} \qquad a_1 v_1 + \cdots + a_k v_k = 0$$

Suppose $k = 1$; then $a_1 v_1 = 0$, $a_1 \neq 0$, and so $v_1 = 0$. But the v_i are nonzero vectors; hence $k > 1$ and $v_k = -a_k^{-1} a_1 v_1 - \cdots - a_k^{-1} a_{k-1} v_{k-1}$. That is, v_k is a linear combination of the preceding vectors.

Theorem 8.2: The nonzero rows $R_1, \ldots, R_n$ of a matrix in echelon form are linearly independent.

8.32 Prove Theorem 8.2.

Suppose $\{R_n, R_{n-1}, \ldots, R_1\}$ is dependent. Then one of the rows, say R_m, is a linear combination of the preceding rows:

$$R_m = a_{m+1} R_{m+1} + a_{m+2} R_{m+2} + \cdots + a_n R_n \qquad (1)$$

Now suppose the kth component of R_m is its first nonzero entry. Then, since the matrix is in echelon form, the kth components of $R_{m+1}, \ldots, R_n$ are all 0, and so the kth component of *(1)* is $a_{m+1} \cdot 0 + a_{m+2} \cdot 0 + \cdots + a_n \cdot 0 = 0$. But this contradicts the assumption that the kth component of R_m is not 0. Thus $R_1, \ldots, R_n$ are independent.

8.33 Suppose $\{v_1, \ldots, v_m\}$ spans a vector space V and suppose $w \in V$. Show that $\{w, v_1, \ldots, v_m\}$ is linearly dependent and spans V.

The vector w is a linear combination of the v_i since $\{v_i\}$ spans V. Accordingly, $\{w, v_1, \ldots, v_m\}$ is linearly dependent. Clearly, w with the v_i span V since the v_i by themselves span V. That is, $\{w, v_1, \ldots, v_m\}$ spans V.

8.34 Suppose $\{v_1, \ldots, v_m\}$ spans a vector space V and suppose v_i is a linear combination of the preceding vectors. Show that $\{v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_m\}$ spans V.

Suppose $v_i = k_1 v_1 + \cdots + k_{i-1} v_{i-1}$. Let $u \in V$. Since $\{v_i\}$ spans V, u is a linear combination of the v_i, say, $u = a_1 v_1 + \cdots + a_m v_m$. Substituting for v_i, we obtain

$$\begin{aligned} u &= a_1 v_1 + \cdots + a_{i-1} v_{i-1} + a_i(k_1 v_1 + \cdots + k_{i-1} v_{i-1}) + a_{i+1} v_{i+1} + \cdots + a_m v_m \\ &= (a_1 + a_i k_1) v_1 + \cdots + (a_{i-1} + a_i k_{i-1}) v_{i-1} + a_{i+1} v_{i+1} + \cdots + a_m v_m \end{aligned}$$

Thus $\{v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_m\}$ spans V. In other words, we can delete v_i from the spanning set and still retain a spanning set.

Lemma 8.3 ("Replacement" Lemma): Suppose $\{v_1, \ldots, v_n\}$ spans a vector space V and $\{w_1, \ldots, w_m\}$ is linearly independent. Then $m \leq n$ and V is spanned by a set of the form $\{w_1, \ldots, w_m, v_{i_1}, \ldots, v_{i_{n-m}}\}$. Thus, in particular, any $n+1$ or more vectors in V are linearly dependent.

8.35 Prove Lemma 8.3.

It suffices to prove the theorem in the case that the v_i are all not 0. (Prove!) Since the $\{v_i\}$ generates V, we have, by Problem 8.33, that

$$\{w_1, v_1, \ldots, v_n\} \tag{1}$$

is linearly dependent and also generates V. By Lemma 8.1, one of the vectors in *(1)* is a linear combination of the preceding vectors. This vector cannot be w_1, so it must be one of the v's, say v_j. Thus by the preceding problem we can delete v_j from the generating set *(1)* and obtain the generating set

$$\{w_1, v_1, \ldots, v_{j-1}, v_{j+1}, \ldots, v_n\} \tag{2}$$

Now we repeat the argument with the vector w_2. That is, since *(2)* generates V, the set

$$\{w_1, w_2, v_1, \ldots, v_{j-1}, v_{j+1}, \ldots, v_n\} \tag{3}$$

is linearly dependent and also generates V. Again by Lemma 8.1, one of the vectors in *(3)* is a linear combination of the preceding vectors. We emphasize that this vector cannot be w_1 or w_2 since $\{w_1, \ldots, w_m\}$ is independent; hence it must be one of the v's, say v_k. Thus by the preceding problem we can delete v_k from the generating set *(3)* and obtain the generating set $\{w_1, w_2, v_1, \ldots, v_{j-1}, v_{j+1}, \ldots, v_{k-1}, v_{k+1}, \ldots, v_n\}$.

We repeat the argument with w_3 and so forth. At each step we are able to add one of the w's and delete one of the v's in the generating set. If $m \le n$, then we finally obtain a generating set of the required form:

$$\{w_1, \ldots, w_m, v_{i_1}, \ldots, v_{i_{n-m}}\}$$

Last, we show that $m > n$ is not possible. Otherwise, after n of the above steps, we obtain the generating set $\{w_1, \ldots, w_n\}$. This implies that w_{n+1} is a linear combination of $w_1, \ldots, w_n$ which contradicts the hypothesis that $\{w_i\}$ is linearly independent.

Theorem 8.4: Let V be a finite-dimensional vector space. Then every basis of V has the same number of vectors.

8.36 Prove Theorem 8.4 (a basic result of linear algebra).

Suppose $\{e_1, e_2, \ldots, e_n\}$ is a basis of V and $\{f_1, f_2, \ldots\}$ is another basis of V. Since $\{e_i\}$ generates V, the basis $\{f_1, f_2, \ldots\}$ must contain n or less vectors, or else it is dependent by the preceding problem. On the other hand, if the basis $\{f_1, f_2, \ldots\}$ contains less than n vectors, then $\{e_1, \ldots, e_n\}$ is dependent by the preceding problem. Thus the basis $\{f_1, f_2, \ldots\}$ contains exactly n vectors, and so the theorem is true.

8.37 Define a maximal independent subset of a set S of vectors in V.

A subset $\{v_1, \ldots, v_m\}$ of S is a maximal independent subset of S if it is independent and if, for any $w \in S$, the set $\{v_1, \ldots, v_m, w\}$ is dependent.

Theorem 8.5: Suppose $\{v_1, \ldots, v_m\}$ is a maximal independent subset of a set S where S spans a vector space V. Then $\{v_1, \ldots, v_m\}$ is a basis of V.

8.38 Prove Theorem 8.5.

Suppose $w \in S$. Then, since $\{v_i\}$ is a maximal independent subset of S, $\{v_1, \ldots, v_m, w\}$ is dependent. Thus w is a linear combination of the v_i, that is, $w \in \text{span}(v_i)$. Hence $S \subseteq \text{span}(v_i)$. This leads to $V = \text{span}(S) \subseteq \text{span}(v_i) \subseteq V$. Thus $\{v_i\}$ spans V and, since it is independent, it is a basis of V.

8.39 Suppose V is generated by a finite set S. Show that V is of finite dimension and, in particular, a subset of S is a basis of V.

Method 1. Of all the independent subsets of S, and there is a finite number of them since S is finite, one of them is maximal. By the preceding problem this subset of S is a basis of V.

Method 2. If S is independent, it is a basis of V. If S is dependent, one of the vectors is a linear combination of the preceding vectors. We may delete this vector and still retain a generating set. We continue this process until we obtain a subset which is independent and generates V, i.e., is a basis of V.

8.40 Consider a finite sequence of vectors $S=\{v_1, v_2, \ldots, v_n\}$. Let T be the sequence of vectors obtained from S by one of the following "elementary operations": (i) interchange two vectors, (ii) multiply a vector by a nonzero scalar, (iii) add a multiple of one vector to another. Show that S and T generate the same space W. Also show that T is independent if and only if S is independent.

▌ Observe that, for each operation, the vectors in T are linear combinations of vectors in S. On the other hand, each operation has an inverse of the same type (Prove!); hence the vectors in S are linear combinations of vectors in T. Thus S and T generate the same space W. Also, T is independent if and only if $\dim W=n$, and this is true iff S is also independent.

8.41 Let $A=(a_{ij})$ and $B=(b_{ij})$ be row equivalent $m\times n$ matrices over a field K, and let $v_1, \ldots, v_n$ be any vectors in a vector space V over K. Let

$$\begin{array}{ll} u_1 = a_{11}v_1 + a_{12}v_2 + \cdots + a_{1n}v_n & w_1 = b_{11}v_1 + b_{12}v_2 + \cdots + b_{1n}v_n \\ u_2 = a_{21}v_1 + a_{22}v_2 + \cdots + a_{2n}v_n & w_2 = b_{21}v_1 + b_{22}v_2 + \cdots + b_{2n}v_n \\ \cdots\cdots\cdots\cdots\cdots\cdots & \cdots\cdots\cdots\cdots\cdots\cdots \\ u_m = a_{m1}v_1 + a_{m2}v_2 + \cdots + a_{mn}v_n & w_m = b_{m1}v_1 + b_{m2}v_2 + \cdots + b_{mn}v_n \end{array}$$

Show that $\{u_i\}$ and $\{w_i\}$ generate the same space.

▌ Applying an "elementary operation" of the preceding problem to $\{u_i\}$ is equivalent to applying an elementary row operation to the matrix A. Since A and B are row equivalent, B can be obtained from A by a sequence of elementary row operations; hence $\{w_i\}$ can be obtained from $\{u_i\}$ by the corresponding sequence of operations. Accordingly, $\{u_i\}$ and $\{w_i\}$ generate the same space.

Theorem 8.6: Let $v_1, \ldots, v_n$ belong to a vector space V over a field K. Let

$$\begin{array}{l} w_1 = a_{11}v_1 + a_{12}v_2 + \cdots + a_{1n}v_n \\ w_2 = a_{21}v_1 + a_{22}v_2 + \cdots + a_{2n}v_n \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ w_n = a_{n1}v_1 + a_{n2}v_2 + \cdots + a_{nn}v_n \end{array}$$

where $a_{ij}\in K$. Let P be the n-square matrix of coefficients, i.e., let $P=(a_{ij})$.
(i) Suppose P is invertible. Then $\{w_i\}$ and $\{v_i\}$ span the same space; hence $\{w_i\}$ is independent if and only if $\{v_i\}$ is independent.
(ii) Suppose P is not invertible. Then $\{w_i\}$ is dependent.
(iii) Suppose $\{w_i\}$ is independent. Then P is invertible.

8.42 Prove (i) of Theorem 8.6: Suppose P is invertible. Then $\operatorname{span}(w_i)=\operatorname{span}(v_i)$; hence $\{w_i\}$ is independent if and only if $\{v_i\}$ is independent.

▌ Since P is invertible, it is row equivalent to the identity matrix I. Hence by the preceding problem $\{w_i\}$ and $\{v_i\}$ generate the same space. Thus one is independent if and only if the other is.

8.43 Prove (ii) of Theorem 8.6: Suppose P is not invertible. Then $\{w_i\}$ is dependent.

▌ Since P is not invertible, it is row equivalent to a matrix with a zero row. This means that $\{w_i\}$ generates a space which has a generating set of less than n elements. Thus $\{w_i\}$ is dependent.

8.44 Prove (iii) of Theorem 8.6: Suppose $\{w_i\}$ is independent. Then P is invertible.

▌ This is the contrapositive of the statement of (ii) and so it follows from (ii).

8.45 Let K be a subfield of a field L and L a subfield of a field E: that is, $K\subset L\subset E$. [Hence K is a subfield of E.] Suppose that E is of dimension n over L and L is of dimension m over K. Show that E is of dimension mn over K.

▮ Suppose $\{v_1, \ldots, v_n\}$ is a basis of E over L and $\{a_1, \ldots, a_m\}$ is a basis of L over K. We claim that $\{a_i v_j: i = 1, \ldots, m, j = 1, \ldots, n\}$ is a basis of E over K. Note that $\{a_i v_j\}$ contains mn elements.

Let w be any arbitrary element in E. Since $\{v_1, \ldots, v_n\}$ generates E over L, w is a linear combination of the v_i with coefficients in L:

$$w = b_1 v_1 + b_2 v_2 + \cdots + b_n v_n \qquad b_i \in L \tag{1}$$

Since $\{a_1, \ldots, a_m\}$ generates L over K, each $b_i \in L$ is a linear combination of the a_j with coefficients in K:

$$\begin{aligned} b_1 &= k_{11}a_1 + k_{12}a_2 + \cdots + k_{1m}a_m \\ b_2 &= k_{21}a_1 + k_{22}a_2 + \cdots + k_{2m}a_m \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ b_n &= k_{n1}a_1 + k_{n2}a_2 + \cdots + k_{nm}a_m \end{aligned}$$

where $k_{ij} \in K$. Substituting in (1), we obtain

$$\begin{aligned} w &= (k_{11}a_1 + \cdots + k_{1m}a_m)v_1 + (k_{21}a_1 + \cdots + k_{2m}a_m)v_2 + \cdots + (k_{n1}a_1 + \cdots + k_{nm}a_m)v_n \\ &= k_{11}a_1v_1 + \cdots + k_{1m}a_mv_1 + k_{21}a_1v_2 + \cdots + k_{2m}a_mv_2 + \cdots + k_{n1}a_1v_n + \cdots + k_{nm}a_mv_n \\ &= \sum_{i,j} k_{ji}(a_iv_j) \end{aligned}$$

where $k_{ji} \in K$. Thus w is a linear combination of the a_iv_j with coefficients in K; hence $\{a_iv_j\}$ generates E over K.

The proof is complete if we show that $\{a_iv_j\}$ is linearly independent over K. Suppose, for scalars $x_{ji} \in K$, $\sum_{i,j} x_{ji}(a_iv_j) = 0$; that is,

$$(x_{11}a_1v_1 + x_{12}a_2v_1 + \cdots + x_{1m}a_mv_1) + \cdots + (x_{n1}a_1v_n + x_{n2}a_2v_n + \cdots + x_{nm}a_mv_n) = 0$$

or

$$(x_{11}a_1 + x_{12}a_2 + \cdots + x_{1m}a_m)v_1 + \cdots + (x_{n1}a_1 + x_{n2}a_2 + \cdots + x_{nm}a_m)v_n = 0$$

Since $\{v_1, \ldots, v_n\}$ is linearly independent over L and since the above coefficients of the v_i belong to L, each coefficient must be 0:

$$x_{11}a_1 + x_{12}a_2 + \cdots + x_{1m}a_m = 0, \quad \ldots, \quad x_{n1}a_1 + x_{n2}a_2 + \cdots + x_{nm}a_n = 0$$

But $\{a_1, \ldots, a_m\}$ is linearly independent over K; hence since the $x_{ji} \in K$,

$$x_{11} = 0, \quad x_{12} = 0, \quad \ldots, \quad x_{1m} = 0, \quad \ldots, \quad x_{n1} = 0, \quad x_{n2} = 0, \quad \ldots, \quad x_{nm} = 0$$

Accordingly, $\{a_iv_j\}$ is linearly independent over K and the theorem is proved.

8.4 BASES AND DIMENSION

8.46 What is meant by the usual basis of the vector space $\mathbf{R}^n$?

▮ Consider the following n vectors in $\mathbf{R}^n$:

$$e_1 = (1, 0, 0, \ldots, 0, 0), \quad e_2 = (0, 1, 0, \ldots, 0, 0), \quad \ldots, \quad e_n = (0, 0, \ldots, 0, 1)$$

These vectors are linearly independent and span $\mathbf{R}^n$. [See Problem 8.49.] Thus the vectors form a basis of $\mathbf{R}^n$ called the *usual basis* of $\mathbf{R}^n$.

8.47 Show that $\dim \mathbf{R}^n = n$.

▮ The above usual basis of $\mathbf{R}^n$ has n vectors; hence $\dim \mathbf{R}^n = n$.

8.48 Let U be the vector space of all 2×3 matrices over a field K. Show that $\dim U = 6$.

▮ The following six matrices,

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

are linearly independent and span U, and hence form a basis of U. [See Problem 8.49.] Thus $\dim U = 6$.

8.49 Let V be the vector space of $m \times n$ matrices over a field K. Let $E_{ij} \in V$ be the matrix with 1 as the ij-entry and 0 elsewhere. Show that $\{E_{ij}\}$ is a basis of V. Thus $\dim V = mn$. [This basis is called the usual basis of V.]

▮ We need to show that $\{E_{ij}\}$ spans V and is independent. Let $A = (a_{ij})$ be any matrix in V. Then $A = \sum_{i,j} a_{ij}E_{ij}$. Hence $\{E_{ij}\}$ spans V.
Now suppose that $\sum_{i,j} x_{ij}E_{ij} = 0$ where the x_{ij} are scalars. The ij-entry of $\sum_{i,j} x_{ij}E_{ij}$ is x_{ij}, and the ij-entry of 0 is 0. Thus $x_{ij} = 0$, $i = 1, \ldots, m$, $j = 1, \ldots, n$. Accordingly the matrices E_{ij} are independent. Thus $\{E_{ij}\}$ is a basis of V.

Remark: Viewing a vector in K^n as a $1 \times n$ matrix, we have shown by the above result that the usual basis of $\mathbf{R}^n$ defined in Problem 8.46 is a basis of $\mathbf{R}^n$.

Theorem 8.7: Suppose $\dim V = n$; say $\{e_1, \ldots, e_n\}$ is a basis of V. Then

(i) Any set of $n + 1$ or more vectors is linearly dependent.
(ii) Any linearly independent set is part of a basis.
(iii) A linearly independent set with n elements is a basis.

8.50 Prove (i) of Theorem 8.7: Any set of $n + 1$ or more vectors is linearly dependent.

▮ Since $\{e_1, \ldots, e_n\}$ generates V, any $n + 1$ or more vectors is dependent by Lemma 8.3.

8.51 Prove (ii) of Theorem 8.7: Any linearly independent set is part of a basis.

▮ Suppose $\{v_1, \ldots, v_r\}$ is independent. By Lemma 8.3, V is generated by a set of the form $S = \{v_1, \ldots, v_r, e_{i_1}, \ldots, e_{i_{n-r}}\}$. By the preceding problem, a subset of S is a basis. But S contains n elements and every basis of V contains n elements. Thus S is a basis of V and contains $\{v_1, \ldots, v_r\}$ as a subset.

8.52 Prove (iii) of Theorem 8.7: A linearly independent set with n elements is a basis.

▮ By (ii), an independent set T with n elements is part of a basis. But every basis of V contains n elements. Thus, T is a basis.

8.53 Show that the following four vectors form a basis of $\mathbf{R}^4$: $(1, 1, 1, 1)$, $(0, 1, 1, 1)$, $(0, 0, 1, 1)$, $(0, 0, 0, 1)$.

▮ The vectors form a matrix in echelon form, and so the vectors are linearly independent. Furthermore, since $\dim \mathbf{R}^4 = 4$, they form a basis of $\mathbf{R}^4$.

8.54 Determine whether or not each of the following form a basis of $\mathbf{R}^3$: (*a*) $(1, 1, 1)$ and $(1, -1, 5)$; (*b*) $(1, 2, 3)$, $(1, 0, -1)$, $(3, -1, 0)$, and $(2, 1, -2)$.

▮ A basis of $\mathbf{R}^3$ must contain exactly three elements, since $\dim \mathbf{R}^3 = 3$. Therefore, neither the vectors in (*a*) nor the vectors in (*b*) form a basis of $\mathbf{R}^3$.

8.55 Determine whether the vectors $(1, 1, 1)$, $(1, 2, 3)$, $(2, -1, 1)$ form a basis of $\mathbf{R}^3$.

▮ The three vectors form a basis if and only if they are independent. Thus form the matrix whose rows are the given vectors and row reduce to echelon form:

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 2 & -1 & 1 \end{pmatrix} \text{ to } \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & -3 & -1 \end{pmatrix} \text{ to } \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 5 \end{pmatrix}$$

The echelon matrix has no zero rows; hence the three vectors are independent and so form a basis for $\mathbf{R}^3$.

8.56 Determine whether $(1, 1, 2)$, $(1, 2, 5)$, $(5, 3, 4)$ form a basis of $\mathbf{R}^3$.

▮ Form the matrix whose rows are the given vectors and row reduce to echelon form:

$$\begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 5 \\ 5 & 3 & 4 \end{pmatrix} \text{ to } \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 3 \\ 0 & -2 & -6 \end{pmatrix} \text{ to } \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{pmatrix}$$

The echelon matrix has a zero row, i.e., only two nonzero rows; hence the three vectors are dependent and so do not form a basis for $\mathbf{R}^3$.

Problems 8.57–8.59 refer to the vector space V of polynomials in t of degree $\le n$.

8.57 Show that $\{1, t, t^2, \ldots, t^n\}$ is a basis of V; hence $\dim V = n + 1$.

Clearly each polynomial in V is a linear combination of $1, t, \ldots, t^{n-1}$ and t^n. Furthermore, $1, t, \ldots, t^{n-1}$ and t^n are independent since none is a linear combination of the preceding polynomials. Thus $\{1, t, \ldots, t^n\}$ is a basis of V.

8.58 Show that $\{1, t-1, (t-1)^2, \ldots, (t-1)^n\}$ is a basis of V.

[Since $\dim V = n+1$, any $n+1$ independent polynomials form a basis of V.] Now each polynomial in the sequence $1, 1-t, \ldots, (1-t)^n$ is of degree higher than the preceding ones and so is not a linear combination of the preceding ones. Thus the $n+1$ polynomials $1, 1-t, \ldots, (1-t)^n$ are independent and so form a basis of V.

8.59 Determine whether or not $\{1+t, t+t^2, t^2+t^3, \ldots, t^{n-1}+t^n\}$ is a basis of V.

The polynomials are linearly independent since each one is of degree higher than the preceding ones. However, the set contains only n elements and $\dim V = n+1$; hence it is not a basis of V.

8.60 Let V be the vector space of 2×2 symmetric matrices over K. Show that $\dim V = 3$. [Recall that $A = (a_{ij})$ is symmetric iff $A = A^T$ or, equivalently, $a_{ij} = a_{ji}$.]

An arbitrary 2×2 symmetric matrix is of the form $A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$ where $a, b, c \in K$. [Note that there are three "variables."] Setting (i) $a=1$, $b=0$, $c=0$; (ii) $a=0$, $b=1$, $c=0$; and (iii) $a=0$, $b=0$, $c=1$, we obtain the respective matrices

$$E_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad E_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad E_3 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

We show that $\{E_1, E_2, E_3\}$ is a basis of V, i.e., that it (1) generates V and (2) is independent.
(1) For the above arbitrary matrix A in V, we have

$$A = \begin{pmatrix} a & b \\ b & c \end{pmatrix} = aE_1 + bE_2 + cE_3$$

Thus $\{E_1, E_2, E_3\}$ generates V.
(2) Suppose $xE_1 + yE_2 + zE_3 = 0$, where x, y, z are unknown scalars. That is, suppose

$$x\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + y\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + z\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} x & y \\ y & z \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Setting corresponding entries equal to each other, we obtain $x=0$, $y=0$, $z=0$. In other words, $xE_1 + yE_2 + zE_3 = 0$ implies $x=0$, $y=0$, $z=0$. Accordingly, $\{E_1, E_2, E_3\}$ is independent.
Thus $\{E_1, E_2, E_3\}$ is a basis of V and so the dimension of V is 3.

8.61 Let W be the vector space of 3×3 symmetric matrices over K. Show that $\dim W = 6$ by exhibiting a basis of W. [Recall that $A = (a_{ij})$ is symmetric iff $a_{ij} = a_{ji}$.]

The following six matrices form a basis of W:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

8.62 What is the dimension of the vector space U of $n\times n$ symmetric matrices over a field K?

As indicated by Problem 8.61, each element on or above the diagonal corresponds to a basis element; hence $\dim U = n + (n-1) + \cdots + 2 + 1 = \frac{1}{2}n(n+1)$.

8.63 Let W be the vector space of 3×3 antisymmetric matrices over K. Show that $\dim W = 3$ by exhibiting a basis of W. [Recall that $A=(a_{ij})$ is antisymmetric iff $a_{ij}=-a_{ji}$.]

The following three matrices form a basis of W:

$$\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

8.64 What is the dimension of the vector space U of $n\times n$ antisymmetric matrices over a field K?

As indicated by Problem 8.63, each element above the diagonal corresponds to a basis element; hence $\dim U=(n-1)+(n-2)+\cdots+2+1=\frac{1}{2}n(n-1)$.

8.65 Show that the complex field **C** is a vector space of dimension 2 over the real field **R**.

We claim that $\{1,i\}$ is a basis of **C** over **R**. For if $v\in\mathbf{C}$, then $v=a+bi=a\cdot 1+b\cdot i$ where $a,b\in\mathbf{R}$; i.e., $\{1,i\}$ generates **C** over **R**. Furthermore, if $x\cdot 1+y\cdot i=0$ or $x+yi=0$, where $x,y\in\mathbf{R}$, then $x=0$ and $y=0$; i.e., $\{1,i\}$ is linearly independent over **R**. Thus $\{1,i\}$ is a basis of **C** over **R**, and so **C** is of dimension 2 over **R**.

8.66 Show that the real field **R** is a vector space of infinite dimension over the rational field **Q**.

We claim that, for any n, $\{1,\pi,\pi^2,\ldots,\pi^n\}$ is linearly independent over **Q**. For suppose $a_0 1+a_1\pi+a_2\pi^2+\cdots+a_n\pi^n=0$, where the $a_i\in\mathbf{Q}$, and not all the a_i are 0. Then π is a root of the following nonzero polynomial over **Q**: $a_0+a_1x+a_2x^2+\cdots+a_nx^n$. But it can be shown that π is a transcendental number, i.e., that π is not a root of any nonzero polynomial over **Q**. Accordingly, the $n+1$ real numbers $1,\pi,\pi^2,\ldots,\pi^n$ are linearly independent over **Q**. Thus for any finite n, **R** cannot be of dimension n over **Q**, i.e., **R** is of infinite dimension over **Q**.

8.67 Let V be the vector space of ordered pairs of complex numbers over the real field **R**. Show that V is of dimension 4.

We claim that the following is a basis of V: $B=\{(1,0),(i,0),(0,1),(0,i)\}$. Suppose $v\in V$. Then $v=(z,w)$ where z,w are complex numbers, and so $v=(a+bi,c+di)$ where a,b,c,d are real numbers. Then $v=a(1,0)+b(i,0)+c(0,1)+d(0,i)$. Thus B generates V.

The proof is complete if we show that B is independent. Suppose $x_1(1,0)+x_2(i,0)+x_3(0,1)+x_4(0,i)=0$ where $x_1,x_2,x_3,x_4\in R$. Then

$$(x_1+x_2i,\,x_3+x_4i)=(0,0) \quad\text{and so}\quad \begin{cases} x_1+x_2i=0 \\ x_3+x_4i=0 \end{cases}$$

Accordingly $x_1=0$, $x_2=0$, $x_3=0$, $x_4=0$ and so B is independent.

8.68 Suppose $\dim V=n$. Show that a generating set with n elements is a basis.

Suppose $u_1,u_2,\ldots,u_n$ span V and the vectors are linearly dependent. Then one of them is a linear combination of the others and so may be deleted from the spanning set. Hence V is spanned by $n-1$ vectors. This is impossible since $\dim V=n$. Thus the u_i are linearly independent and hence form a basis of V.

8.5 DIMENSION AND SUBSPACES

Theorem 8.8: Let W be a subspace of an n-dimensional vector space V. Then $\dim W\le n$. In particular, if $\dim W=n$, then $W=V$.

8.69 Prove Theorem 8.8 which gives the basic relationship between the dimension of a vector space V and the dimension of a subspace W of V.

Since V is of dimension n, any $n+1$ or more vectors are linearly dependent. Furthermore, since a basis of W consists of linearly independent vectors, it cannot contain more than n elements. Accordingly, $\dim W\le n$.