CHAPTER 10
Linear Mappings

All vector spaces are assumed to be over the same field K unless otherwise stated or implied.

10.1 LINEAR MAPPINGS

10.1 Define a linear mapping.

▮ Let V and U be vector spaces over the same field K. A mapping $F: V \to U$ is called a *linear mapping* [or *linear transformation* or *vector space homomorphism*] if it satisfies the following two conditions:
(1) For any $v, w \in V$, $F(v+w) = F(v) + F(w)$.
(2) For any $k \in K$ and any $v \in V$, $F(kv) = kF(v)$.
In other words, $F: V \to U$ is linear if it "preserves" the two basic operations of a vector space, that of vector addition and that of scalar multiplication.

10.2 Suppose $F: V \to U$ is linear. Show that $F(0) = 0$.

▮ Substitute $k = 0$ into $F(kv) = kF(v)$ to get $F(0) = 0$.

10.3 Suppose $F: V \to U$ is linear. Show that $F(-u) = -F(u)$.

▮ Using $F(ku) = kF(u)$, we have $F(-u) = F[(-1)u] = (-1)F(u) = -F(u)$.

10.4 Show that $F: V \to U$ is linear if and only if, for any scalars $a, b \in K$ and any vectors $v, w \in V$,

$$F(av + bw) = F(av) + F(bw) = aF(v) + bF(w) \qquad (1)$$

▮ Suppose F is linear, then $F(av + bw) = F(av) + F(bw) = aF(v) + bF(w)$. Conversely, suppose (1) holds. For $a = 1$ and $b = 1$ we get $F(v+w) = F(v) + F(w)$, and for $b = 0$ we get $F(av) = aF(v)$; hence F is linear.

Remark: The condition $F(av + bw) = aF(v) + bF(w)$ completely characterizes linear mappings and is sometimes used as its definition.

10.5 Suppose $F: V \to U$ is linear. Show that, for any $a_i \in K$ and any $v_i \in V$,

$$F(a_1v_1 + a_2v_2 + \cdots + a_nv_n) = a_1F(v_1) + a_2F(v_2) + \cdots + a_nF(v_n)$$

▮ Since F is linear, the condition holds for $n = 1$ and $n = 2$. Suppose $n > 2$. Then, by induction, $F(a_1v_1 + a_2v_2 + \cdots + a_nv_n) = F(a_1v_1) + F(a_2v_2 + \cdots + a_nv_n) = a_1F(v_1) + a_2F(v_2) + \cdots + a_nF(v_n)$. [This condition will be used frequently throughout the text.]

10.6 Let A be any $m \times n$ matrix over a field K. As noted previously, A determines a mapping $T: K^n \to K^m$ by the assignment $v \mapsto Av$. [Here the vectors in K^n in K^m are written as columns.] Show that T is linear.

▮ By properties of matrices, $T(v+w) = A(v+w) = Av + Aw = T(v) + T(w)$ and $T(kv) = A(kv) = kAv = kT(v)$, where $v, w \in K^n$ and $k \in K$. Thus T is linear.

Remark: The above type of linear mapping shall occur again and again. In fact, in the next chapter we show that every linear mapping from one finite-dimensional vector space into another can be represented as a linear mapping of the above type.

10.7 Let $F: \mathbf{R}^3 \to \mathbf{R}^3$ be the "projection" mapping into the xy plane, i.e., $F(x, y, z) = (x, y, 0)$. Show that F is linear.

Let $v=(a,b,c)$ and $w=(a',b',c')$. Then

$$F(v+w)=F(a+a',\ b+b',\ c+c')=(a+a',\ b+b',\ 0)=(a,b,0)+(a',b',0)=F(v)+F(w)$$

and, for any $k\in\mathbf{R}$, $F(kv)=F(ka,kb,kc)=(ka,kb,0)=k(a,b,0)=kF(v)$. That is, F is linear.

10.8 Let $F:\mathbf{R}^2\to\mathbf{R}^2$ be the "translation" mapping defined by $F(x,y)=(x+1,\ y+2)$. Show that F is not linear.

Observe that $F(0)=F(0,0)=(1,2)\neq 0$. That is, the zero vector is not mapped onto the zero vector. Hence F is not linear.

10.9 Let $F:V\to U$ be the mapping which assigns $0\in U$ to every $v\in V$. Show that F is linear.

For any $v,w\in V$ and any $k\in K$, $F(v+w)=0=0+0=F(v)+F(w)$ and $F(kv)=0=k0=kF(v)$. Thus F is linear. We call F the *zero mapping* and shall usually denote it by 0.

10.10 Consider the identity mapping $I:V\to V$ which maps each $v\in V$ into itself. Show that I is linear.

For any $v,w\in V$ and any $a,b\in K$, we have $I(av+bw)=av+bw=aI(v)+bI(w)$. Thus I is linear.

Problems 10.11–10.12 refer to the vector space V of polynomials in the variable t over the real field $\mathbf{R}$.

10.11 Let $D:V\to V$ be the differential mapping $D(v)=dv/dt$. Show that D is linear.

It is proven in calculus that

$$\frac{d(u+v)}{dt}=\frac{du}{dt}+\frac{dv}{dt}\quad\text{and}\quad\frac{d(ku)}{dt}=k\,\frac{du}{dt}$$

i.e., $D(u+v)=D(u)+D(v)$ and $D(ku)=kD(u)$. Thus D is linear.

10.12 Let $I:V\to\mathbf{R}$ be the integral mapping $I(v)=\int_0^1 v(t)\,dt$. Show that I is linear.

It is proven in calculus that

$$\int_0^1 (u(t)+v(t))dt=\int_0^1 u(t)dt+\int_0^1 v(t)dt$$

and

$$\int_0^1 ku(t)dt=k\int_0^1 u(t)dt$$

i.e., $I(u+v)=I(u)+I(v)$ and $I(ku)=kI(u)$. Thus I is linear.

10.13 Consider the mapping $F:\mathbf{R}^2\to\mathbf{R}^2$ defined by $F(x,y)=(x+y,x)$. Show that F is linear.

Let $v=(a,b)$ and $w=(a',b')$; hence $v+w=(a+a',b+b')$ and $kv=(ka,kb)$. We have $F(v)=(a+b,a)$ and $F(w)=(a'+b',a')$. Thus

$$F(v+w)=F(a+a',b+b')=(a+a'+b+b',\ a+a')=(a+b,a)+(a'+b',a')=F(v)+F(w)$$

and $F(kv)=F(ka,kb)=(ka+kb,ka)=k(a+b,a)=kF(v)$. Since v, w, and k were arbitrary, F is linear.

10.14 Consider $F:\mathbf{R}^3\to\mathbf{R}$ defined by $F(x,y,z)=2x-3y+4z$. Show that F is linear.

Let $v=(a,b,c)$ and $w=(a',b',c')$; hence

$$v+w=(a+a',\ b+b',\ c+c')\quad\text{and}\quad kv=(ka,kb,kc)\qquad k\in\mathbf{R}$$

We have $F(v)=2a-3b+4c$ and $F(w)=2a'-3b'+4c'$. Thus $F(v+w)=F(a+a',\ b+b',\ c+c')=2(a+a')-3(b+b')+4(c+c')=(2a-3b+4c)+(2a'-3b'+4c')=F(v)+F(w)$ and $F(kv)=F(ka,kb,kc)=2ka-3kb+4kc=k(2a-3b+4c)=kF(v)$. Accordingly, F is linear.

10.15 Consider $F: \mathbf{R}^2 \to \mathbf{R}$ defined by $F(x, y) = xy$. Show that F is not linear.

▌ Let $v = (1, 2)$ and $w = (3, 4)$; then $v + w = (4, 6)$. We have $F(v) = 1 \cdot 2 = 2$ and $F(w) = 3 \cdot 4 = 12$. Hence $F(v + w) = F(4, 6) = 4 \cdot 6 = 24 \neq F(v) + F(w)$. Accordingly, F is not linear.

10.16 Consider $F: \mathbf{R}^2 \to \mathbf{R}^3$ defined by $F(x, y) = (x + 1, 2y, x + y)$. Show that F is not linear.

▌ Since $F(0, 0) = (1, 0, 0) \neq (0, 0, 0)$, F cannot be linear.

10.17 Consider $F: \mathbf{R}^3 \to \mathbf{R}^2$ defined by $F(x, y, z) = (|x|, 0)$. Show that F is not linear.

▌ Let $v = (1, 2, 3)$ and $k = -3$; hence $kv = (-3, -6, -9)$. We have $F(v) = (1, 0)$ and so $kF(v) = -3(1, 0) = (-3, 0)$. Then $F(kv) = F(-3, -6, -9) = (3, 0) \neq kF(v)$ and hence F is not linear.

10.18 Consider $F: \mathbf{R}^2 \to \mathbf{R}^2$ defined by $F(x, y) = (2x - y, x)$. Show that F is linear.

▌ Let $u = (a, b)$ and $v = (a', b')$. Then $u + v = (a + a', b + b')$ and $k(u) = (ka, kb)$. We have $F(u) = (2a - b, a)$ and $F(v) = (2a' - b', a')$. Thus

$$F(u + v) = F(a + a', b + b') = [2(a + a') - (b + b'),\ a + a'] = (2a - b, a) + (2a' - b', a') = F(u) + F(v)$$

and $F(ku) = F(ka, kb) = (2ka - kb, ka) = k(2a - b, a) = kF(u)$. Thus F is linear.

10.19 Consider $F: \mathbf{R}^2 \to \mathbf{R}$ defined by $F(t) = (2t, 3t)$. Show that F is linear.

▌
$$F(t_1 + t_2) = [2(t_1 + t_2), 3(t_1 + t_2)] = [2t_1 + 2t_2, 3t_1 + 3t_2] = (2t_1, 3t_1) + (2t_2, 3t_2) = F(t_1) + F(t_2)$$

and

$$F(kt) = (2kt, 3kt) = k(2t, 3t) = kF(t)$$

Thus F is linear.

10.20 Consider $F: \mathbf{R}^2 \to \mathbf{R}^2$ defined by $F(x, y) = (x^2, y^2)$. Show that F is not linear.

▌ Let $u = (1, 2)$ and $k = 3$. Then $ku = (3, 6)$. We have $F(u) = (1, 4)$ and so $kF(u) = (3, 12)$. Hence $F(ku) = F(3, 6) = (9, 16) \neq kF(u)$. Thus F is not linear.

10.21 Consider $F: \mathbf{R}^3 \to \mathbf{R}^2$ defined by $F(x, y, z) = (x + 1, y + z)$. Show that F is not linear.

▌ $F(0) = F(0, 0, 0) = (0 + 1, 0 + 0) = (1, 0) \neq (0, 0)$. Thus F is not linear.

10.22 Consider $F: \mathbf{R}^2 \to \mathbf{R}$ defined by $F(x, y) = |x + y|$. Show that F is not linear.

▌ Let $u = (1, 2)$ and $k = -3$; so $ku = (-3, -6)$. We have $F(u) = 1 + 2 = 3$, hence $kF(u) = (-3) \cdot 3 = -9$. Thus $F(ku) = F(-3, -6) = -3 - 6 = -9 = 9 \neq kF(u)$. Accordingly, F is not linear.

Problems 10.23–10.25 refer to the vector space V of n-square matrices over a field K and an arbitrary matrix M in V.

10.23 Let $T: V \to V$ be defined by $T(A) = AM + MA$, where $A \in V$. Show that T is linear.

▌ For any $A, B \in V$ and any $k \in K$, we have $T(A + B) = (A + B)M + M(A + B) = AM + BM + MA + MB = (AM + MA) + (BM + MB) = T(A) + T(B)$ and $T(kA) = (kA)M + M(kA) = k(AM) + k(MA) = k(AM + MA) = kT(A)$. Accordingly, T is linear.

10.24 Let $T: V \to V$ be defined by $T(A) = M + A$ where $A \in V$. Show that T is linear if and only if $M = 0$.

▌ If $M = 0$, then $T(A) = A$, that is, T is the identity map; hence T is linear. On the other hand, suppose $M \neq 0$. Then $T(0) = M + 0 = M \neq 0$ and so T is not linear.

10.25 Let $T: V \to V$ be defined by $T(A) = MA$ where $A \in V$. Show that T is linear.

For any $A, B \in V$ and any $a, b \in K$, we have $T(aA + bB) = M(aA + bB) = aMA + bMB = aT(A) + bT(B)$. Thus T is linear.

10.26 Let V be the vector space of polynomials in t over K. Show that the mapping $T: V \to V$ is linear where $T(a_0 + a_1 t + \cdots + a_n t^n) = a_0 t + a_1 t^2 + \cdots + a_n t^{n+1}$.

Note T multiplies a polynomial $f(t)$ by t, that is, $T(f(t)) = tf(t)$. Hence $T(f(t) + g(t)) = t(f(t) + g(t)) = tf(t) + tg(t) = T(f(t)) + T(g(t))$ and, for any scalar $k \in K$, $T(kf(t)) = t(kf(t)) = k(tf(t)) = kT(f(t))$. Thus T is linear.

Problems 10.27–10.28 refer to the conjugate mapping $T: \mathbf{C} \to \mathbf{C}$ on the complex field $\mathbf{C}$. That is, $T(z) = \bar{z}$ where $z \in \mathbf{C}$, or $T(a + bi) = a - bi$ where $a, b \in \mathbf{R}$.

10.27 Show that T is not linear if $\mathbf{C}$ is viewed as a vector space over itself.

Let $u = 3 + 4i$ and $k = 2 - i$. Then $ku = (2 - i)(3 + 4i) = 10 + 5i$ and $T(ku) = 10 - 5i$. However, $kT(u) = (2 - i)(3 - 4i) = 2 - 11i \neq T(ku)$. Thus T is not linear.

10.28 Show that T is linear if $\mathbf{C}$ is viewed as a vector space over the real field $\mathbf{R}$.

Let $z = a + bi$ and $w = c + di$ where $a, b, c, d \in \mathbf{R}$. Then $z + w = (a + c) + (b + d)i$. Then $T(z + w) = (a + c) - (b + d)i = (a - bi) + (c - di) = T(z) + T(w)$. Also, for $k \in \mathbf{R}$, we have $kz = ka + kbi$. Hence $T(kz) = ka - kbi = k(a - bi) = kT(z)$. Thus T is linear.

10.2 PROPERTIES OF LINEAR MAPPINGS

Theorem 10.1: Let V and U be vector spaces over a field K. Let $\{v_1, \ldots, v_n\}$ be a basis of V and let $u_1, \ldots, u_n$ be any arbitrary vectors in U. Then there exists a unique linear mapping $F: V \to U$ such that $F(v_1) = u_1, F(v_2) = u_2, \ldots, F(v_n) = u_n$.

This section uses the Theorem 10.1 whose proof appears in Problems 10.43–10.45.

10.29 Show that there is a unique linear map $F: \mathbf{R}^2 \to \mathbf{R}^2$ for which $F(1, 2) = (2, 3)$ and $F(0, 1) = (1, 4)$.

Since $(1, 2)$ and $(0, 1)$ form a basis of $\mathbf{R}^2$, such a linear map F exists and is unique by Theorem 10.1.

Problems 10.30–10.32 refer to the linear map F in Problem 10.29.

10.30 Find a formula for F, i.e., find $F(a, b)$.

Write (a, b) as a linear combination of $(1, 2)$ and $(0, 1)$ using unknowns x and y:

$$(a, b) = x(1, 2) + y(0, 1) = (x, 2x + y) \qquad \text{so} \qquad a = x,\ b = 2x + y$$

Solve for x and y in terms of a and b to get $x = a$, $y = -2a + b$. Then $F(a, b) = xF(1, 2) + yF(0, 1) = a(2, 3) + (-2a + b)(1, 4) = (b, -5a + 4b)$.

10.31 Find $F(5, 6)$.

Use the formula for F to get $F(5, 6) = (6, -25 + 24) = (6, -1)$.

10.32 Find $F^{-1}(-2, 7)$.

Set $F(a, b) = (-2, 7)$ and solve for a and b. We get $(b, -5a + 4b) = (-2, 7)$ so $b = -2$ and $-5a + 4b = 7$. Then $a = -3$, $b = -2$. Thus $F^{-1}(-2, 7) = (-3, -2)$.

10.33 Show there is a unique linear map $T: \mathbf{R}^2 \to \mathbf{R}^2$ for which $T(3,1) = (2,-4)$ and $T(1,1) = (0,2)$.

Since $(3,1)$ and $(1,1)$ are linearly independent, they form a basis for $\mathbf{R}^2$; hence such a linear map T exists and is unique by Theorem 10.1.

Problems 10.34–10.36 refer to the linear map T in Problem 10.33.

10.34 Find a formula for T.

First write (a,b) as a linear combination of $(3,1)$ and $(1,1)$ using unknown scalars x and y:

$$(a,b) = x(3,1) + y(1,1)$$

Hence

$$(a,b) = (3x,x) + (y,y) = (3x+y, x+y) \qquad \text{and so} \qquad \begin{cases} 3x + y = a \\ x + y = b \end{cases}$$

Solving for x and y in terms of a and b, $x = \frac{1}{2}a - \frac{1}{2}b$ and $y = -\frac{1}{2}a + \frac{3}{2}b$. Therefore, $T(a,b) = xT(3,1) + yT(1,1) = x(2,-4) + y(0,2) = (2x,-4x) + (0,2y) = (2x, -4x+2y) = (a-b, 5b-3a)$.

10.35 Find $T(7,4)$.

Use the formula for T to get $T(7,4) = (7-4, 20-21) = (3,-1)$.

10.36 Find $T^{-1}(5,-3)$.

Set $T(a,b) = (5,-3)$ and solve for a and b. We get $(a-b, -3a+5b) = (5,-3)$ so $a-b=5$, $-3a+5b=-3$. Then $a=11$, $b=6$. Thus $F^{-1}(5,-3) = (11,6)$.

10.37 Show there is a unique linear map $T: \mathbf{R}^2 \to \mathbf{R}$ for which $T(1,1) = 3$ and $T(0,1) = -2$.

Since $\{(1,1),(0,1)\}$ is a basis of $\mathbf{R}^2$, such a linear mapping exists and is unique by Theorem 10.1.

Problems 10.38–10.41 refer to the linear map T in Problem 10.37.

10.38 Find a formula for T.

First we write (a,b) as a linear combination of $(1,1)$ and $(0,1)$ using unknown scalars x and y:

$$(a,b) = x(1,1) + y(0,1)$$

Then $(a,b) = (x,x) + (0,y) = (x, x+y)$ and so $x = a$, $x+y = b$. Solving for x and y in terms of a and b, we obtain $x = a$ and $y = b-a$. Therefore, $T(a,b) = T(x(1,1) + y(0,1)) = xT(1,1) + yT(0,1) = a(3) + (b-a)(-2) = 5a - 2b$.

10.39 Find $T(8,2)$ and $T(-4,6)$.

Use the formula for T to get $T(8,2) = 40 - 4 = 36$ and $T(-4,6) = -20 - 12 = -32$.

10.40 Find $T^{-1}(6)$.

Set $T(a,b) = 6$ to get $5a - 2b = 6$. Here b is a free variable. Set $b = t$ where t is a parameter to get the solution $a = (2t+6)/5$, $b = t$. Thus $T^{-1}(6) = \{((2t+6)/5, t) : t \in \mathbf{R}\}$.

10.41 Is T one-to-one?

No, since $T^{-1}(6)$ has more than one element, e.g., $T(\frac{6}{5}, 0) = 6$ and $T(\frac{8}{5}, 0) = 6$.

10.42 Is there a linear map $T: \mathbf{R}^2 \to \mathbf{R}^2$ for which $T(2,2) = (8,-6)$ and $T(5,5) = (3,-2)$?

Theorem 10.1 does not apply here since $(2,2)$ and $(5,5)$ are linearly dependent and hence do not form a basis of $\mathbf{R}^2$. Note $(5,5) = \frac{5}{2}(2,2)$. If T is linear then $T(5,5) = T(\frac{5}{2}(2,2)) = \frac{5}{2}T(2,2) = \frac{5}{2}(8,-6) = (20,-15)$. But $T(5,5) = (3,-2)$ is given. Thus such a linear map T does not exist.

Problems 10.43–10.45 refer to the proof of Theorem 10.1 which consists of three steps:
(1) Define the mapping $F: V \to U$ such that $F(v_i) = u_i$, $i = 1, \ldots, n$.
(2) Show that F is linear.
(3) Show that F is unique.

10.43 Step (1): Define the required map $F: V \to U$ such that $F(v_i) = u_i$.

Let $v \in V$. Since $\{v_1, \ldots, v_n\}$ is a basis of V, there exist unique scalars $a_1, \ldots, a_n \in K$ for which $v = a_1v_1 + a_2v_2 + \cdots + a_nv_n$. We define $F: V \to U$ by $F(v) = a_1u_1 + a_2u_2 + \cdots + a_nu_n$. [Since the a_i are unique, the mapping F is well-defined.] Now, for $i = 1, \ldots, n$, $v_i = 0v_1 + \cdots + 1v_i + \cdots + 0v_n$. Hence $F(v_i) = 0u_1 + \cdots + 1u_i + \cdots + 0u_n = u_i$. Thus the first step of the proof is complete.

10.44 Step (2): Show F is linear.

Suppose $v = a_1v_1 + a_2v_2 + \cdots + a_nv_n$ and $w = b_1v_1 + b_2v_2 + \cdots + b_nv_n$. Then $v + w = (a_1 + b_1)v_1 + (a_2 + b_2)v_2 + \cdots + (a_n + b_n)v_n$ and, for any $K \in K$, $kv = ka_1v_1 + ka_2v_2 + \cdots + ka_nv_n$. By definition of the mapping F, $F(v) = a_1u_1 + a_2u_2 + \cdots + a_nu_n$ and $F(w) = b_1u_1 + b_2u_2 + \cdots + b_nu_n$. Hence

$$\begin{aligned} F(v + w) &= (a_1 + b_1)u_1 + (a_2 + b_2)u_2 + \cdots + (a_n + b_n)u_n \\ &= (a_1u_1 + a_2u_2 + \cdots + a_nu_n) + (b_1u_1 + b_2u_2 + \cdots + b_nu_n) \\ &= F(v) + F(w) \end{aligned}$$

and

$$F(k) = k(a_1u_1 + a_2u_2 + \cdots + a_nu_n) = kF(v)$$

Thus F is linear.

10.45 Step (3): Show F is unique.

Suppose $G: V \to U$ is linear and $G(v_i) = u_i$, $i = 1, \ldots, n$. If $v = a_1v_1 + a_2v_2 + \cdots + a_nv_n$, then

$$\begin{aligned} G(v) &= G(a_1v_1 + a_2v_2 + \cdots + a_nv_n) = a_1G(v_1) + a_2G(v_2) + \cdots + a_nG(v_n) \\ &= a_1u_1 + a_2u_2 + \cdots + a_nu_n = F(v) \end{aligned}$$

Since $G(v) = F(v)$ for every $v \in V$, $G = F$. Thus F is unique and the theorem is proved.

10.46 Suppose the linear mapping $F: V \to U$ is one-to-one and onto. Show that the inverse mapping $F^{-1}: U \to V$ is also linear.

Suppose $u, u' \in U$. Since F is one-to-one and onto, there exist unique vectors $v, v' \in V$ for which $F(v) = u$ and $F(v') = u'$. Since F is linear, we also have $F(v + v') = F(v) + F(v') = u + u'$ and $F(kv) = kF(v) = ku$. By definition of the inverse mapping, $F^{-1}(u) = v$, $F^{-1}(u') = v'$, $F^{-1}(u + u') = v + v'$, and $F^{-1}(ku) = kv$. Then $F^{-1}(u + u') = v + v' = F^{-1}(u) + F^{-1}(u')$ and $F^{-1}(ku) = kv = kF^{-1}(u)$ and thus F^{-1} is linear.

10.47 Suppose $F: V \to U$ and $G: U \to W$ are linear mappings. Show that the composition mapping $G \circ F: V \to W$ is linear. [Recall that $G \circ F$ is defined by $(G \circ F)(v) = G(F(v))$.]

For any vectors $v, w \in V$ and any scalars $a, b \in K$, $(G \circ F)(av + bw) = G(F(av + bw)) = G(aF(v) + bF(w)) = aG(F(v)) + bG(F(w)) = a(G \circ F)(v) + b(G \circ F)(w)$. Thus $G \circ F$ is linear.

10.48 Let $\{e_1, e_2, e_3\}$ be a basis of V and $\{f_1, f_2\}$ a basis of U. Let $T: V \to U$ be linear. Furthermore, suppose

$$\begin{aligned} T(e_1) &= a_1 f_1 + a_2 f_2 \\ T(e_2) &= b_1 f_1 + b_2 f_2 \\ T(e_3) &= c_1 f_1 + c_2 f_2 \end{aligned} \quad \text{and} \quad A = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{pmatrix}$$

Show that, for any $v \in V$, $A[v]_e = [T(v)]_f$ where the vectors in K^2 and K^3 are written as column vectors.

▮ Suppose $v = k_1 e_1 + k_2 e_2 + k_3 e_3$; then $[v]_e = \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix}$. Also,

$$\begin{aligned} T(v) &= k_1 T(e_1) + k_2 T(e_2) + k_3 T(e_3) \\ &= k_1(a_1 f_1 + a_2 f_2) + k_2(b_1 f_1 + b_2 f_2) + k_3(c_1 f_1 + c_2 f_2) \\ &= (a_1 k_1 + b_1 k_2 + c_1 k_3) f_1 + (a_2 k_1 + b_2 k_2 + c_2 k_3) f_2 \end{aligned}$$

Accordingly,

$$[T(v)]_f = \begin{pmatrix} a_1 k_1 + b_1 k_2 + c_1 k_3 \\ a_2 k_1 + b_2 k_2 + c_2 k_3 \end{pmatrix}$$

Computing, we obtain

$$A[v]_e = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} = \begin{pmatrix} a_1 k_1 + b_1 k_2 + c_1 k_3 \\ a_2 k_1 + b_2 k_2 + c_2 k_3 \end{pmatrix} = [T(v)]_f$$

10.49 Let $T: V \to U$ be linear, and suppose $v_1, \ldots, v_n \in V$ have the property that their images $T(v_1), \ldots, T(v_n)$ are linearly independent. Show that the vectors $v_1, \ldots, v_n$ are also linearly independent.

▮ Suppose that, for scalars $a_1, \ldots, a_n$, $a_1 v_1 + a_2 v_2 + \cdots + a_n v_n = 0$. Then

$$0 = T(0) = T(a_1 v_1 + a_2 v_2 + \cdots + a_n v_n) = a_1 T(v_1) + a_2 T(v_2) + \cdots + a_n T(v_n)$$

Since the $T(v_i)$ are linearly independent, all the $a_i = 0$. Thus $v_1, \ldots, v_n$ are linearly independent.

10.3 KERNEL AND IMAGE OF A LINEAR MAPPING

10.50 Let $F: V \to U$ be a linear mapping. Define the kernel of F.

▮ The *kernel* of F, written Ker F, is the set of elements in V which map into $0 \in U$:

$$\text{Ker } F = \{v \in V: F(v) = 0\}$$

10.51 Let $F: V \to U$ be a linear mapping. Define the image of F.

▮ The *image* of F, written Im F, is the set of image points in U:

$$\text{Im } F = \{u \in U: \exists v \in V \text{ for which } F(v) = u\}$$

10.52 Let $F: \mathbf{R}^3 \to \mathbf{R}^3$ be the projection mapping into the xy plane, i.e., defined by $F(x, y, z) = (x, y, 0)$. Find the kernel of F.

▮ The points on the z axis, and only these points, map into the zero vector $0 = (0, 0, 0)$. Thus Ker $F = \{(0, 0, c): c \in \mathbf{R}\}$.

10.53 Find the image of the projection mapping $F(x, y, z) = (x, y, 0)$ in Problem 10.52.

▮ The image of F consists precisely of those points in the xy plane: Im $F = \{(a, b, 0): a, b \in \mathbf{R}\}$.

10.54 Let $F: \mathbf{R}^3 \to \mathbf{R}^3$ be the linear mapping which rotates a vector about the z axis through an angle θ:

$$F(x, y, z) = (x \cos\theta - y \sin\theta, x \sin\theta + y \cos\theta, z)$$

Find the kernel of F.

▮ Under a rotation, the length of a vector does not change. Thus only the zero vector is mapped into the zero vector; hence $\text{Ker } F = \{0\}$. [In other words, setting $F(x, y, z) = (0, 0, 0)$ yields only $x = 0$, $y = 0$, $z = 0$.]

10.55 Find the image of the rotation map F in Problem 10.54.

▮ Since one can always rotate back by an angle $-\theta$, every vector $v \in \mathbf{R}^3$ is in the image of F; that is, $\text{Im } F = \mathbf{R}^3$.

Problems 10.56–10.60 refer to the vector space V of real polynomials in the variable t and the third derivative map $D^3: V \to V$, that is, $D^3(f) = d^3f/dt^3$. [Frequently, one uses D for the first derivative, D^2 for the second derivative, and so on.]

10.56 Find $D^3(f)$ where $f(t) = t^4 - 2t^3 + 5t^2 - 6t + 9$.

▮ Take the derivative three times:

$$\frac{df}{dt} = 4t^3 - 6t^2 + 10t - 6 \qquad \frac{d^2f}{dt^2} = 12t^2 - 12t + 10 \qquad D^3(f) = \frac{d^3f}{dt^3} = 24t - 12$$

10.57 Find $D^3(g)$ where $g(t) = at^2 + bt + c$.

▮
$$\frac{dg}{dt} = 2at + b \qquad \frac{d^2g}{dt^2} = 2a \qquad D^3(g) = \frac{d^3g}{dt^3} = 0$$

10.58 Find the kernel of D^3.

▮ The third derivative of any polynomial of degree two or less equals zero and those of higher degree are not zero. Thus $\text{Ker } D^3 = \{f \in V: \deg f \le 2\}$.

10.59 Find the preimage of $h(t) = t^3$ [denoted by $D^{-3}(h)$].

▮ Integrate three times:

$$D^{-1}(h) = \frac{t^4}{4} + C_1 \qquad D^{-2}(h) = \frac{t^5}{20} + C_1t + C_2 \qquad D^{-3}(h) = \frac{t^6}{120} + \frac{C_1t^2}{2} + C_2t + C_3 = \frac{t^6}{120} + at^2 + bt + c$$

10.60 Find the image of D^3.

▮ Given any polynomial $f(t)$, one can integrate three times to obtain a polynomial $F(t)$ such that d^3F/dt^3 yields $f(t)$. Thus the image of D^3 contains every polynomial $f(t)$, that is, $\text{Im } D^3 = V$.

10.61 Suppose $F: V \to U$ is a linear mapping. Show that the kernel of F is a subspace of V.

▮ Since $F(0) = 0$, $0 \in \text{Ker } F$. Now suppose $v, w \in \text{Ker } F$ and $a, b \in K$. Since v and w belong to the kernel of F, $F(v) = 0$ and $F(w) = 0$. Thus $F(av + bw) = aF(v) + bF(w) = a0 + b0 = 0$ and so $av + bw \in \text{Ker} F$. Thus the kernel of F is a subspace of V.

10.62 Suppose $F: V \to U$ is a linear mapping. Show that the image of F is a subspace of U.

▮ Since $F(0) = 0$, $0 \in \text{Im } F$. Now suppose $u, u' \in \text{Im } F$ and $a, b \in K$. Since u and u' belong to the image of F, there exist vectors $v, v' \in V$ such that $F(v) = u$ and $F(v') = u'$. Then $F(av + bv') = aF(v) + bF(v') = au + bu' \in \text{Im } F$. Thus the image of F is a subspace of U.

10.63 Suppose the vectors $v_1, \ldots, v_n$ span V and that $F: V \to U$ is linear. Show that the vectors $F(v_1), \ldots, F(v_n) \in U$ span $\text{Im } F$.

▮ Suppose $u \in \operatorname{Im} F$; then $F(v) = u$ for some vector $v \in V$. Since $v_1, \ldots, v_n$ span V and since $v \in V$, there exist scalars $a_1, \ldots, a_n$ for which $v = a_1v_1 + a_2v_2 + \cdots + a_nv_n$. Accordingly,

$$u = F(v) = F(a_1v_1 + a_2v_2 + \cdots + a_nv_n) = a_1F(v_1) + a_2F(v_2) + \cdots + a_nF(v_n)$$

Thus the vectors $F(v_1), \ldots, F(v_n)$ span $\operatorname{Im} F$.

10.64 Let $A = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \\ d_1 & d_2 & d_3 \end{pmatrix}$ be an arbitrary 4×3 matrix over a field K. [Recall that we view A as a linear mapping $A: K^3 \to K^4$.] Show that the image of A is precisely the column space of A.

▮ Let e_1, e_2, e_3 be the usual basis vectors of K^3. Since e_1, e_2, e_3 span K^3, their values Ae_1, Ae_2, Ae_3 under A span the image of A. But the vectors Ae_1, Ae_2, and Ae_3 are the columns of A:

$$Ae_1 = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \\ d_1 & d_2 & d_3 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} a_1 \\ b_1 \\ c_1 \\ d_1 \end{pmatrix} \qquad Ae_2 = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \\ d_1 & d_2 & d_3 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a_2 \\ b_2 \\ c_2 \\ d_2 \end{pmatrix} \qquad Ae_3 = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \\ d_1 & d_2 & d_3 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} a_3 \\ b_3 \\ c_3 \\ d_3 \end{pmatrix}$$

Thus the image of A is precisely the column space of A.

Remark: We emphasize that if A is any $m \times n$ matrix over a field K, then we view A as a linear map $A: K^n \to K^m$ where vectors are written as columns. In such a case, the image of A is the column space of A. On the other hand, some texts view the matrix A as a linear map $A: K^m \to K^n$ where vectors are written as rows, and there the image of A is the row space of A.

10.65 Suppose V has finite dimension and $F: V \to U$ is linear. Show that $\operatorname{Im} F$ has finite dimension, and $\dim(\operatorname{Im} F) \le \dim V$.

▮ Suppose $\dim V = n$ and $\dim(\operatorname{Im} F) > \dim V$. Then there exist vectors $w_1, w_2, \ldots, w_{n+1} \in \operatorname{Im} F$ which are linearly independent. Let $v_1, v_2, \ldots, v_{n+1}$ be vectors in V such that $F(v_i) = w_i$. Suppose $a_1v_1 + \cdots + a_{n+1}v_{n+1} = 0$. Then $0 = F(0) = F(a_1v_1 + \cdots + a_{n+1}v_{n+1}) = a_1F(v_1) + \cdots + a_{n+1}F(v_{n+1}) = a_1w_1 + \cdots + a_{n+1}w_{n+1}$. Since the w_i are linearly independent, $a_1 = 0, \ldots, a_{n+1} = 0$. Thus $v_1, v_2, \ldots, v_{n+1}$ are linearly independent. This contradicts the fact that $\dim V = n$. Thus $\dim(\operatorname{Im} F) \le \dim V$.

Theorem 10.2: Let V be of finite dimension and let $F: V \to U$ be a linear mapping. Then

$$\dim V = \dim(\operatorname{Ker} F) + \dim(\operatorname{Im} F)$$

[That is, the sum of the dimensions of the image and kernel of a linear mapping is equal to the dimension of its domain.]

10.66 Prove Theorem 10.2.

▮ Suppose $\dim(\operatorname{Ker} F) = r$ and $\{w_1, \ldots, w_r\}$ is a basis of $\operatorname{Ker} F$, and suppose $\dim(\operatorname{Im} F) = s$ and $\{u_1, \ldots, u_s\}$ is a basis of $\operatorname{Im} F$. [By Problem 10.65, $\operatorname{Im} F$ has finite dimension.] Since $u_j \in \operatorname{Im} F$, there exist vectors $v_1, \ldots, v_s$ in V such that $F(v_1) = u_1, \ldots, F(v_s) = u_s$. We claim that the set $B = \{w_1, \ldots, w_r, v_1, \ldots, v_s\}$ is a basis of V, i.e., (i) B spans V and (ii) B is linearly independent. Once we prove (i) and (ii), then $\dim V = r + s = \dim(\operatorname{Ker} F) + \dim(\operatorname{Im} F)$.

(i) B spans V. Let $v \in V$. Then $F(v) \in \operatorname{Im} F$. Since the u_j span $\operatorname{Im} F$, there exist scalars $a_1, \ldots, a_s$ such that $F(v) = a_1u_1 + \cdots + a_su_s$. Set $\hat{v} = a_1v_1 + \cdots + a_sv_s - v$. Then $F(\hat{v}) = F(a_1v_1 + \cdots + a_sv_s - v) = a_1F(v_1) + \cdots + a_sF(v_s) - F(v) = a_1u_1 + \cdots + a_su_s - F(v) = 0$. Thus $\hat{v} \in \operatorname{Ker} F$. Since the w_i span $\operatorname{Ker} F$, there exist scalars $b_1, \ldots, b_r$ such that $\hat{v} = b_1w_1 + \cdots + b_rw_r = a_1v_1 + \cdots + a_sv_s - v$. Accordingly, $v = a_1v_1 + \cdots + a_sv_s - b_1w_1 - \cdots - b_rw_r$. Thus B spans V.

(ii) B is linearly independent. Suppose

$$x_1w_1 + \cdots + x_rw_r + y_1v_1 + \cdots + y_sv_s = 0 \qquad (1)$$

where the $x_i, y_j \in K$. Then

$$0 = F(0) = F(x_1w_1 + \cdots + x_rw_r + y_1v_1 + \cdots + y_sv_s) = x_1F(w_1) + \cdots + x_rF(w_r) + y_1F(v_1) + \cdots + y_sF(v_s) \quad (2)$$

But $F(w_i) = 0$ since $w_i \in \operatorname{Ker} F$ and $F(v_j) = u_j$. Substitution in *(2)* gives $y_1u_1 + \cdots + y_su_s = 0$. Since the u_j are linearly independent, each $y_j = 0$. Substitution in *(1)* gives $x_1w_1 + \cdots + x_rw_r = 0$. Since the w_i are linearly independent, each $x_i = 0$. Thus B is linearly independent.

10.67 Define the rank of a linear map $F: V \to U$.

I The *rank* of F is defined to be the dimension of its image, i.e., $\operatorname{rank}(F) = \dim(\operatorname{Im} F)$.

10.68 Define the nullity of a linear map $F: V \to U$.

I The *nullity* of F is defined to be the dimension of its kernel, i.e., $\operatorname{nullity}(F) = \dim(\operatorname{Ker} F)$.

10.69 Restate Theorem 10.2 using the above terminology.

I ***Theorem 10.2:*** Let $F: V \to U$ be linear where V has finite dimension. Then $\operatorname{rank}(F) + \operatorname{nullity}(F) = \dim(\operatorname{Dom} F)$ [where Dom F denotes the domain V of F].

10.70 The rank of a matrix A was originally defined to be the dimension of its column space and of its row space. How is this definition related to the definition of rank in Problem 10.67?

I Both definitions give the same value since the image of A is precisely its column space.

10.71 Let $F: V \to U$ and $G: U \to W$ be linear. Show that $\operatorname{rank}(G \circ F) \le \operatorname{rank} G$.

I Since $F(V) \subset U$, we also have $G(F(V)) \subset G(U)$ and so $\dim G(F(V)) \le \dim G(U)$. Then $\operatorname{rank}(G \circ F) = \dim((G \circ F)(V)) = \dim(G(F(V))) \le \dim G(U) = \operatorname{rank} G$.

10.72 Let $F: V \to U$ and $G: U \to W$ be linear. Show that $\operatorname{rank}(G \circ F) \le \operatorname{rank} F$.

I We have $\dim(G(F(V))) \le \dim F(V)$. Hence $\operatorname{rank}(G \circ F) = \dim((G \circ F)(V)) = \dim(G(F(V))) \le \dim F(V) = \operatorname{rank} F$.

10.73 Suppose $f: V \to U$ is linear with kernel W and $f(v) = u$. Show that $v + W = \{v + w: w \in W\}$ is the preimage of u, that is, $f^{-1}(u) = v + W$. [The set $v + W$ is called a coset of W.]

I We must prove that (i) $f^{-1}(u) \subset v + W$ and (ii) $v + W \subset f^{-1}(u)$. We first prove (i). Suppose $v' \in f^{-1}(u)$. Then $f(v') = u$ and so $f(v' - v) = f(v') - f(v) = u - u = 0$, that is, $v' - v \in W$. Thus $v' = v + (v' - v) \in v + W$ and hence $f^{-1}(u) \subset v + W$.

Now we prove (ii). Suppose $v' \in v + W$. Then $v' = v + w$ where $w \in W$. Since W is the kernel of f, $f(w) = 0$. Accordingly, $f(v') = f(v + w) = f(v) + f(w) = f(v) + 0 = f(v) = u$. Thus $v' \in f^{-1}(u)$ and so $v + W \subset f^{-1}(u)$.

10.4 COMPUTING THE KERNEL AND IMAGE OF LINEAR MAPPINGS

10.74 Let $F: \mathbf{R}^4 \to \mathbf{R}^3$ be the linear mapping defined by $F(x, y, s, t) = (x - y + s + t, x + 2s - t, x + y + 3s - 3t)$. Find a basis and the dimension of the image U of F.

I Find the image of the usual basis vectors of $\mathbf{R}^4$:

$$F(1,0,0,0) = (1,1,1) \qquad F(0,1,0,0) = (-1,0,1) \qquad F(0,0,1,0) = (1,2,3)$$
$$F(0,0,0,1) = (1,-1,-3)$$

The image vectors span U; hence form the matrix whose rows are these image vectors and row reduce to echelon form:

$$\begin{pmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 1 & 2 & 3 \\ 1 & -1 & -3 \end{pmatrix} \text{ to } \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \\ 0 & -2 & -4 \end{pmatrix} \text{ to } \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Thus $\{(1,1,1),(0,1,2)\}$ is a basis of U; hence $\dim U = 2$.

10.75 Find a basis and the dimension of the kernel W of the map F in Problem 10.74.

I Set $F(v)=0$ where $v=(x,y,z,t)$:

$$F(x,y,s,t)=(x-y+s+t,\ x+2s-t,\ x+y+3s-3t)=(0,0,0)$$

Set corresponding components equal to each other to form the following homogeneous system whose solution space is the kernel W of F:

$$\begin{aligned} x-y+s+t&=0 \\ x\quad +2s-t&=0 \\ x+y+3s-3t&=0 \end{aligned} \quad\text{or}\quad \begin{aligned} x-y+s+t&=0 \\ y+s-2t&=0 \\ 2y+2s-4t&=0 \end{aligned} \quad\text{or}\quad \begin{aligned} x-y+s+t&=0 \\ y+s-2t&=0 \end{aligned}$$

The free variables are s and t; hence $\dim W=2$. Set
(*a*) $s=-1$, $t=0$ to obtain the solution $(2,1,-1,0)$
(*b*) $s=0$, $t=1$, to obtain the solution $(1,2,0,1)$
Thus $\{(2,1,-1,0),(1,2,0,1)\}$ is a basis of W. [Observe that $\dim U+\dim W=2+2=4$, which is the dimension of the domain $\mathbf{R}^4$ of F.]

10.76 Let $T:\mathbf{R}^3\to\mathbf{R}^3$ be the linear mapping defined by $T(x,y,z)=(x+2y-z,\ y+z,\ x+y-2z)$. Find a basis and the dimension of the image U of T.

I Find the image of vectors which span the domain $\mathbf{R}^3$:

$$T(1,0,0)=(1,0,1) \qquad T(0,1,0)=(2,1,1) \qquad T(0,0,1)=(-1,1,-2)$$

The images span the image U of T; hence form the matrix whose rows are the image vectors and row reduce to echelon form:

$$\begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ -1 & 1 & -2 \end{pmatrix} \text{ to } \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{pmatrix} \text{ to } \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

Thus $\{(1,0,1),(0,1,-1)\}$ is a basis of U, and so $\dim U=2$.

10.77 Find a basis and the dimension of the kernel W of the map T in Problem 10.76.

I Set $T(v)=0$ where $v=(x,y,z)$: $T(x,y,z)=(x+2y-z,\ y+z,\ x+y-2z)=(0,0,0)$. Set corresponding components equal to each other to form the homogeneous system whose solution space is the kernel W of T:

$$\begin{aligned} x+2y-z&=0 \\ y+z&=0 \\ x+y-2z&=0 \end{aligned} \quad\text{or}\quad \begin{aligned} x+2y-z&=0 \\ y+z&=0 \\ -y-z&=0 \end{aligned} \quad\text{or}\quad \begin{aligned} x+2y-z&=0 \\ y+z&=0 \end{aligned}$$

The only free variable is z; hence $\dim W=1$. Let $z=1$; then $y=-1$ and $x=3$. Thus $\{(3,-1,1)\}$ is a basis of W. [Observe that $\dim U+\dim W=2+1=3$, which is the dimension of the domain $\mathbf{R}^3$ of T.]

10.78 Let $F:\mathbf{R}^3\to\mathbf{R}^4$ be defined by $F(x,y,z)=(x+y+z,\ x+2y-3z,\ 2x+3y-2z,\ 3x+4y-z)$. Find a basis and the dimension of the image of F.

I First find the image of vectors which span the domain $\mathbf{R}^3$ of F:

$$F(1,0,0)=(1,1,2,3) \qquad F(0,1,0)=(1,2,3,4) \qquad F(0,0,1)=(1,-3,-2,-1)$$