week 9, Homework, 23.5. 2026.

TECHNICAL TRANSLATION.

1.CANTOR'S THEOREM (PLAIN NOTEPAD VERSION)

1. SUMMARY

Cantor's theorem:

For every set S:

|S| < |P(S)|

meaning:

- there is an injection: S → P(S)
- but there is NO surjection: S → P(S)

So the power set P(S) is always strictly larger than S.

This applies to ALL sets (finite and infinite).

Key idea:

no function f: S → P(S) can be onto.

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2. **BASIC DEFINITIONS**

Power set:

P(S) = set of all subsets of S

Example:
If S = {a, b}

then:
P(S) = { ∅, {a}, {b}, {a,b} }

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3. MAIN THEOREM (Cantor)

There is no surjection:

f: S → P(S)

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4. PROOF (DIAGONAL ARGUMENT)

Assume (for contradiction):

there exists a surjection f: S → P(S)

Define a special subset D ⊆ S:

D = { x in S | x NOT in f(x) }

Now ask:

Is D in the image of f?

If f(a) = D for some a in S, then:

Case 1:
a ∈ D
→ by definition of D:
a ∉ f(a) = D
contradiction

Case 2:
a ∉ D
→ by definition of D:
a ∈ f(a) = D
contradiction

Both cases contradict.

So f cannot be surjective.

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5. CONCLUSION

No surjection exists:

S → P(S)

Therefore:

|S| < |P(S)|

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6. IMPORTANT INTERPRETATION

This means:

- infinity is not unique size
- there are strictly larger infinities
- power set operation always increases cardinality

THEOREM (Lebesgue covering lemma) (SOURCE NOTE
This text is taken and reformatted from:
https://en.wikipedia.org/wiki/Lebesgue%27s_number_lemma)

If (X, d) is compact and U is an open cover of X,
then there exists delta > 0 such that delta is a Lebesgue number of U.

DIRECT PROOF

Let U be an open cover of X.

Since X is compact, there exists a finite subcover:

A1, A2, ..., An ⊆ U such that
X = A1 ∪ A2 ∪ ... ∪ An.

If one of Ai equals X, then any delta > 0 works.

Assume no Ai equals X.

Define Ci = X \ Ai (complement of Ai).
Then each Ci is nonempty.

Define function f: X → R by:

f(x) = (1/n) * sum_{i=1 to n} d(x, Ci)

where d(x, Ci) = inf{ d(x, y) : y in Ci }.

Since X is compact and f is continuous,
f attains a minimum value delta:

delta = min_{x in X} f(x)

We claim delta > 0.

KEY IDEA:

For every x in X, x belongs to some Ai,
so x is NOT in Ci for at least one i,
therefore distance from x to Ci is positive in that index,
so f(x) cannot be zero.

Hence delta > 0.

Now take any subset Y ⊆ X such that:

diam(Y) < delta

Pick any point x0 in Y.

Then for all x in Y:

d(x, x0) < delta

so Y ⊆ B_delta(x0) (ball of radius delta around x0).

Since f(x0) ≥ delta,
there exists some i such that:

d(x0, Ci) ≥ delta

This implies:

the ball B_delta(x0) does NOT intersect Ci,
so:

B_delta(x0) ⊆ Ai

Therefore:

Y ⊆ Ai

CONCLUSION:

Every set of diameter < delta is contained in some Ai,
so delta is a Lebesgue number.

QED

PROOF BY CONTRADICTION (SKETCH)

Assume no Lebesgue number exists.

Then for every $1/k$, there exists a set A_k such that:

$diam(A_k) < 1/k$
but A_k is not contained in any U in the cover.

Pick x_k in A_k.

Since X is sequentially compact,
there exists a subsequence x_{nk} converging to x_0.

Since U is an open cover, x_0 is in some U_{α}.

So there exists $r > 0$ such that:

$B_r(x_0) \subseteq U_{\alpha}$

For large k:
$x_{nk} \to x_0$ implies $x_{nk} \in B_{r/2}(x_0)$

Also $diam(A_{nk}) \to 0$ implies A_{nk} is very small.

Hence for large k:

$A_{nk} \subseteq B_r(x_0) \subseteq U_{\alpha}$

Contradiction.

Therefore a Lebesgue number exists.

SQUEEZE THEOREM (SANDWICH THEOREM)

1. SEQUENCES OF REAL NUMBERS

Let x_n, y_n, z_n be sequences in R.

Assume:

$\lim_{n\to\infty} y_n = L$
$\lim_{n\to\infty} z_n = L$

and for all $n \in N$:

$y_n \leq x_n \leq z_n$

Then:

$\lim_{n\to\infty} x_n = L$

Meaning:

if x_n is always between y_n and z_n,
and both y_n and z_n converge to same limit L,
then x_n also converges to L.

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2. SEQUENCES OF COMPLEX NUMBERS

Let an be a sequence in R such that:

$a_n \to 0$

Let zn be a sequence in C.

Assume:

$|z_n| \leq a_n$ for all $n \in N$

Then:

$z_n \to 0$

(i.e. zn is a null sequence)

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3. LINEARLY ORDERED SPACE VERSION

Let $(S, \leq)$ be a linearly ordered space.

Let xn, yn, zn be sequences in S.

Let $p \in S$.

Assume:

$x_n \to p$
$z_n \to p$

and for all $n \in N$:

$x_n \leq y_n \leq z_n$

Then:

$y_n \to p$

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4. METRIC SPACE VERSION

Let (S, d) be a metric space.

Let $p \in S$.

Let rn be a sequence in R such that:

$r_n \to 0$

Let xn be a sequence in S such that:

$d(x_n, p) \leq r_n$ for all $n \in \mathbb{N}$

Then:

$x_n \to p$

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5. FUNCTION VERSION

Let I be an open interval in R.

Let f(x), g(x), h(x) be functions defined on I except possibly at a point a.

Assume:

$g(x) \leq f(x) \leq h(x)$ for all $x \neq a$

and:

$\lim_{x \to a} g(x) = L$
$\lim_{x \to a} h(x) = L$

Then:

$\lim_{x \to a} f(x) = L$

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6. INTERPRETATION

If a function or sequence is "squeezed" between two others
that converge to the same limit,
then it must also converge to that same limit.

LITERARY TRANSLATIONS

SOURCE:
Excerpt from "Forbidden Knowledge" by Kathryn Krammer

Henley had always been the kind to take input and feedback
over suggestions and wise advice.

Since nothing she ever had to say came through the proper channels,
he had learned to ignore her until she got it right.

This much he had explained to her on one of those rare occasions
when she forced him to talk about his feelings.

That particular night he sat quietly until she was done screaming
and then said:

"Trust is good; control is better,"

and went off to bed.

The Executive Committee meeting the next morning
was held in closed session,
so she wasn't allowed in.

But she was sure Henley was behind all this.

And now Phelony stood before the Ambassador,
armed only with:

- a deeply held ideological and personal commitment to guilt,
- a profound belief in her own worthlessness,
- and a rap — a canned speech —
  which she had carefully memorized before coming
  to argue with the Ambassador about Ext.

For her, Ext had a personal and intimate meaning
that she had never dared discuss with anyone.

Ext was a form of fear.

It was fear within a relationship that,
if the relationship is to survive,
one must cause to vanish.

The Executive Committee had assigned her
to persuade Ambassador Notsniw,
by whatever means necessary,
to denounce publicly the Notsniw Law,
the Ambassador's brainchild,
what holovision editorialists called
his legislative masterpiece.

Editorialists said it was the law
that would protect the public
from forbidden knowledge from the Worm Planet,

protect the public from perverse concepts
that creep into the brain,
which turn ordinarily social human beings
into something else entirely.

The law made publication or possession
of materials pertaining to Ext
a federal offense,

carrying penalties of:

- fifty thousand Erdmarks
  OR
- ten years in prison without hope of parole.

The mathematical techniques for gaining access
to the powers of Ext
had been learned by xenobiologists
while studying the Group Mind on the Worm Planet

and were now in use
by three-quarters of the human population
in both inhabited solar systems

as a method of eliminating emotional problems
by translating them into the external world.

The law was therefore in effect
in both inhabited solar systems.

The pop-psychology tape which popularized Ext
was based upon the case
of a little boy named Winston Notsniw,

son of the cook
on the xenobiological expedition,

who was miraculously cured of his autism
when the Group Mind revealed to him
the geometricity of coiled gastropod shells.