OMEWORK, WEEK 7, 12.5.2026

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1. TRANSLATION TASK (HEINE–CANTOR THEOREM)

Translate the following theorem and proof into Serbian. Preserve the mathematical notation and explain the logical structure of the argument clearly and precisely.

Source of the proof:
• PlanetMath — Proof of Heine–Cantor Theorem
• ProofWiki — Heine–Cantor Theorem

Heine–Cantor Theorem

If K is a compact metric space and
f : K -> X
is continuous, then f is uniformly continuous.

PROOF

Let
eps > 0.

By continuity of f, for every point k in K there exists a number
delta_k > 0
such that
d(k,x) < delta_k
implies
d(f(k),f(x)) < eps/2.

Consider the open balls
B(k, delta_k/2).

These balls cover K.

Since K is compact, there exists a finite subcover:
B(k1, delta_k1/2), B(k2, delta_k2/2), ... , B(kn, delta_kn/2).

Define
delta = min{ delta_k1/2, delta_k2/2, ... , delta_kn/2 }.

We will show that this delta works for uniform continuity.

Let x,y be in K such that
d(x,y) < delta.

Since the finite collection of balls covers K, the point x belongs to one of them:
B(ki, delta_ki/2).

Hence
d(x,ki) < delta_ki/2.

By the triangle inequality:
d(y,ki) <= d(y,x) + d(x,ki)
< delta + delta_ki/2
<= delta_ki.

Therefore
d(f(x),f(ki)) < eps/2
and
d(f(y),f(ki)) < eps/2.

Again by the triangle inequality:
d(f(x),f(y))
<= d(f(x),f(ki)) + d(f(ki),f(y))
< eps/2 + eps/2 = eps.

Hence
d(x,y) < delta => d(f(x),f(y)) < eps.

Therefore f is uniformly continuous.

End of proof.

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2. TAYLOR’S THEOREM (KANE, Writing Proofs in Analysis, pp. 263–264)

8.5.4 Taylor’s Theorem

If
f(x) = SUM_{n=0 to infinity} an(x - c)^n
for all x with |x - c| < R, then f(c) = a0, the constant term of the series for f.

Finding the mth derivative of the series for f and evaluating it at the center c gives:

f^(m)(c) = m! am.

So for all integers m >= 0:
am = f^(m)(c) / m!

This gives a straightforward way to generate the power series representing any analytic function.

Moreover, even if f is not infinitely differentiable, if it is m times differentiable, one can generate the mth degree Taylor polynomial:

g(x) = SUM_{n=0 to m} f^(n)(c)/n! (x - c)^n

Then g is an mth degree polynomial that equals f at c, and all derivatives up to order m agree at c.

The first degree Taylor polynomial is the linear approximation (tangent line at c).

However, this does not guarantee that g(x) approximates f(x) well when x is far from c.

This is addressed by Taylor’s Theorem:

f(x) = g(x) + Rm(x)

where Rm(x) is the remainder term.

Lagrange form:
If f is (m+1)-times differentiable between c and x, then:

Rm(x) = f^(m+1)(ξ) * (x - c)^(m+1) / (m+1)!

for some ξ between c and x.

HIGHER ORDER ROLLE’S THEOREM

Let f be (m+1)-times differentiable on (a,b), continuous on [a,b], and suppose:
0 = f(a) = f'(a) = f''(a) = ... = f^(m)(a) = f(b).

Then there exists x in (a,b) such that:
f^(m+1)(x) = 0.

Proof idea:

• Apply Rolle’s theorem repeatedly
• Use induction on derivatives
• Conclude existence of xm+1

TAYLOR’S THEOREM PROOF STRUCTURE

Let f be (m+1)-times differentiable between c and x.

Define:

$$g(y) = \sum_{n=0}^{m} \frac{f^{(n)}(c)}{n!} (y - c)^n$$

Then:
f(y) - g(y) has zeros of order m at c.

Construct:

$$h(y) = f(y) - g(y) - (f(x) - g(x)) * (y - c)^{m+1} / (x - c)^{m+1}$$

Then:

$$h(c) = h'(c) = \ldots = h^{(m)}(c) = h(x) = 0$$

By higher-order Rolle:
exists ξ between c and x such that:

$$h^{(m+1)}(\xi) = 0$$

This gives:

$$f(x) = \sum_{n=0}^{m} \frac{f^{(n)}(c)}{n!} (x - c)^n + f^{(m+1)}(\xi)(x - c)^{m+1}/(m+1)!$$

EXAMPLE (COSINE)

$$\cos x = 1 - x^2/2! + x^4/4! - x^6/6! + \ldots$$

Approximation at $x = 2$:

$$1 - x^2/2! + x^4/4!$$

Error bound:

$$|R| \le 64/720 \approx 0.09$$

Actual:
cos(2) ≈ -0.416146
polynomial = -1/3
error ≈ 0.08281

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3. TRANSLATION TASK (MARTIN GARDNER TEXT)

Translate the following passage into Serbian.

Text Source:
Excerpt from "Left or Right" by Martin Gardner.

TEXT:

He spread his arms in a hopeless gesture. The sky behind him was a deep scarlet now from the setting suns, and two long shadows branched out from his feet along the dark rocky surface.

"We'll have to hope for the best. Even if we are reversed—and we'll find out soon enough when we reach a charted region—there isn't any way we can somersault back again without running the risk of smashing the ship. And it's too late to go back for another shipment."

Suddenly a thought struck me.

"We ejected a load of garbage and wastepaper just before the crash. If we could locate it and thaw it out, we could find those notes you made, and—"

I didn't have to finish. Karston was whacking me on the back of my space jacket and telling me I wasn't as stupid as he suspected.

We finished the repairs and shoved off the following day. After returning to the vicinity of the crash, it didn't take our radar division long to locate the batch of refuse we had left suspended in space. The material was close to absolute zero, so it was hours after we retrieved it before it had warmed enough to be examined.

Major Karston picked up one of the wadded balls of moist yellow paper. They were notes he had made for a speech in which he planned to announce the successful completion of our mission.

With trembling fingers he smoothed the paper flat and I bent over his shoulder to look. I think my heart was pounding louder than our atomic motors. Would we be able to read it? Or would it be in mirror writing, like the "Jabberwocky" in that ancient classic about the little girl who walked through the looking glass?

Odd or even, left or right, a fifty-fifty chance like the flip of a coin—but the fate of our people was hanging in the balance!