

Homework 7.5. 2026,

NESTED INTERVAL THEOREM (<https://planetmath.org/nestedintervaltheorem>)

Proposition 1

If

$$[a_1, b_1] \supseteq [a_2, b_2] \supseteq [a_3, b_3] \supseteq \dots$$

is a sequence of nested closed intervals, then

$\bigcap_{n=1}^{\infty} [a_n, b_n] \neq \emptyset$.

If also

$$\lim_{n \rightarrow \infty} (b_n - a_n) = 0,$$

then the infinite intersection consists of a unique real number.

PROOF

There are two consequences of nesting intervals:

$[a_m, b_m] \subseteq [a_n, b_n]$ for $n \leq m$.

1.

$a_n \leq a_m$ for $n \leq m$, so the sequence (a_n) is nondecreasing.

2.

We also have:

$a_m \leq b_n$ and $a_n \leq b_m$, hence $a_n \leq b_i$ for all i, n .

So (a_n) is bounded above by every b_i .

Therefore, the limit of (a_n) exists and equals

$$a = \sup(a_n).$$

Similarly, the sequence (b_n) is nonincreasing and bounded below by all a_n , so it has an infimum

$$b = \inf(b_n).$$

Now:

Since $a = \sup(a_n)$, we have

$a_n \leq a \leq b_n$ for all n .

Thus every interval $[a_n, b_n]$ contains $[a, b]$,
so the intersection contains $[a, b]$, hence it is non-empty.

Uniqueness argument:

If c is outside $[a, b]$, say $c < a$,
then by definition of supremum there exists some a_n such that $c < a_n$,
so c is not in $[a_n, b_n]$.

Hence c is not in the intersection.

So the intersection is exactly $[a, b]$.

Now assume:

$\lim (b_n - a_n) = 0$.

Then:

$$\begin{aligned} b - a &= \lim b_n - \lim a_n \\ &= \lim (b_n - a_n) \\ &= 0 \end{aligned}$$

so $a = b$.

Therefore the intersection contains exactly one point.

CONCLUSION

$\bigcap_{n=1}^{\infty} [a_n, b_n] = \{a\}$

REMARK

This theorem is called the nested interval theorem.
It is equivalent to the finite intersection property for compact sets in \mathbb{R} .

Any number between $\sup(a_n)$ and $\inf(b_n)$ lies in all intervals.

PROOF: A continuous function on a closed bounded interval is uniformly continuous

- Let $a < b$ and let f be continuous on $[a,b]$.
- Let $\epsilon > 0$ be given.
- For each x in $[a,b]$, there exists $\delta_x > 0$ such that for all y in $[a,b]$, if $|y - x| < \delta_x$ then $|f(y) - f(x)| < \epsilon/2$.
- The intervals $(x - \delta_x, x + \delta_x)$ form an open cover of $[a,b]$.
- By Heine-Borel theorem, there exist finitely many points x_1, \dots, x_n such that $[a,b]$ is contained in the union of $(x_j - \delta_{x_j}, x_j + \delta_{x_j})$.
- Let E be the set of all endpoints $x_j \pm \delta_{x_j}$.
- Since E is finite, define δ as the minimum positive distance between distinct points in E .
- Let y, z be in $[a,b]$ such that $|y - z| < \delta$.
- Then y and z must lie in the same interval $(x_m - \delta_{x_m}, x_m + \delta_{x_m})$ for some m .
-

So:

$$\begin{aligned} & |f(y) - f(z)| \\ \leq & |f(y) - f(x_m)| + |f(z) - f(x_m)| \end{aligned}$$

- Each term is $< \epsilon/2$, so:
 $|f(y) - f(z)| < \epsilon$.
- Therefore f is uniformly continuous on $[a,b]$. (taken from Kane's Writing proofs in analysis)

ROLLE'S THEOREM

PROOF

Let $a < b$, and let f be a function such that:

- f is continuous on $[a, b]$
- f is differentiable on (a, b)
- $f(a) = f(b)$

We prove that there exists $c \in (a, b)$ such that $f'(c) = 0$.

1.

Since f is continuous on the closed interval $[a, b]$, by the Extreme Value Theorem, f attains both a maximum and a minimum on $[a, b]$.

2.

Case 1: Maximum and minimum occur only at endpoints.

If both extrema occur at a and b , and $f(a) = f(b)$, then the maximum and minimum values are equal.

Hence f is constant on $[a, b]$.

Therefore:

$f'(c) = 0$ for all $c \in (a, b)$.

3.

Case 2: At least one extremum occurs in the interior.

If either maximum or minimum occurs at some $c \in (a, b)$, then c is an interior extremum point.

By Fermat's theorem:
if f has a local extremum at c and is differentiable at c , then

$f'(c) = 0$.

4.

Conclusion:

In both cases, there exists $c \in (a, b)$ such that

$f'(c) = 0$.

This completes the proof.

PROOF: A function differentiable on an interval has the intermediate value property for its derivative

Let f be a function differentiable at every point of an interval I .

Let $a, b \in I$, and assume that $f'(a) \neq f'(b)$.

Without loss of generality, assume:
 $a < b$ and $f'(a) < f'(b)$.

Let K be a number such that:
 $f'(a) < K < f'(b)$.

Define:

$$g(x) = f(x) - Kx$$

Then:

$$g'(x) = f'(x) - K \text{ for all } x \in (a, b)$$

Hence:

$$g'(a) < 0 < g'(b)$$

Since f is differentiable, g is differentiable on (a, b) ,
and therefore g is continuous on $[a, b]$.

TEXT

I took up the receiver to get in immediate touch with the company. It was only then I realized that apart from the address the advertisement gave no particulars. A computer center not on the telephone! It just didn't make sense. I rang up the editors.

"Sorry, but that was all we received from Kraftstuds," the secretary told me. "There was no telephone in the ad."

The Kraftstuds and Co. was not in the telephone directory either.

Burning with impatience I waited for Monday. Whenever I looked up from those neatly penned equations concealing complicated physical processes, my thoughts would turn to Kraftstuds and Co. Men of vision, I thought. In our time and age when mankind endeavors to clothe its every idea in mathematical garbs, it would be hard to imagine a more profitable occupation.

Incidentally, who was this Kraftstuds? I had been resident in the town quite a long time but the name rang no bell. As a matter of fact, I did vaguely recollect having heard the name before. But I couldn't remember when or where, no matter how hard I jogged my memory.

Came Monday. Pocketing the sheet of equations, I started out in search of 12 Weltstrasse. A fine drizzle forced me to take a taxi.

"It's a goodish way off," said the cabby, "beyond the river, next door to the lunatic asylum."

I nodded and off we went.

It took us about forty minutes. We passed through the town gates, went over a bridge, skirted a lake, and found ourselves in the country. Early green shoots could be seen here and there in the fields along the unmetalled road, and the car stalled between banks of mud every now and then, its back wheels skidding furiously.

Then roofs appeared, then the red brick walls of the lunatic asylum standing in a little depression and jocularly referred to in town as the Wise Men's Home.

Along the tall brick wall bristling with bits of broken glass ran a clinker lane. After a few turnings the taxi pulled up at an inconspicuous door.

"This is Number Twelve."

(an excerpt taken from the story "The Maxwell Equations")