

Technical translation.

1.

PROOF: Every convergent sequence is Cauchy

Let (a_j) be a sequence of real numbers such that $\lim_{j \rightarrow \infty} a_j = L$.

Let $\varepsilon > 0$ be given.

By the definition of limit, there exists N such that for all $j > N$, $|a_j - L| < \varepsilon/2$.

Now let $m, n > N$.

Then:

$$|a_m - L| < \varepsilon/2 \text{ and } |a_n - L| < \varepsilon/2.$$

We

estimate:

$$|a_m - a_n|$$

$$= |(a_m - L) + (L - a_n)|$$

$$\leq |a_m - L| + |a_n - L|$$

$$< \varepsilon/2 + \varepsilon/2$$

$$= \varepsilon.$$

Thus, for all $m, n > N$, we have:

$$|a_m - a_n| < \varepsilon.$$

Therefore, the sequence (a_j) is Cauchy. ✓

Note: The converse is also true (every Cauchy sequence in \mathbb{R} converges

2. Assume that (f_n) is a sequence of functions continuous on the interval $[a, b]$ that converges

monotonically to a function f which is also continuous on $[a, b]$. We prove that f_n converges uniformly to f on $[a, b]$.

Let $\varepsilon > 0$ be given.
Let $x \in [a, b]$.

Since f is continuous at x , there exists $\delta_1 > 0$ such that if $y \in [a, b]$ and $|y - x| < \delta_1$, then $|f(y) - f(x)| < \varepsilon/3$.

Since $f_n(x) \rightarrow f(x)$, there exists an integer N_x such that for all $n \geq N_x$, $|f_n(x) - f(x)| < \varepsilon/3$.

Since f_{N_x} is continuous at x , there exists $\delta_2 > 0$ such that if $y \in [a, b]$ and $|y - x| < \delta_2$, then $|f_{N_x}(y) - f_{N_x}(x)| < \varepsilon/3$.

Let $\delta_x = \min(\delta_1, \delta_2)$.

Then if $|y - x| < \delta_x$, we have:
 $|f_{N_x}(y) - f(y)|$
 $\leq |f_{N_x}(y) - f_{N_x}(x)| + |f_{N_x}(x) - f(x)| + |f(x) - f(y)|$
 $< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon$.

The intervals $(x - \delta_x, x + \delta_x)$ cover $[a, b]$.

By the Heine–Borel theorem, there exist finitely many points x_1, x_2, \dots, x_k such that the intervals $(x_j - \delta_{x_j}, x_j + \delta_{x_j})$ cover $[a, b]$.

Let $N = \max(N_{x_1}, N_{x_2}, \dots, N_{x_k})$.

Let $y \in [a, b]$. Then y lies in some interval $(x_j - \delta_{x_j}, x_j + \delta_{x_j})$.

Since the sequence is monotone and $n \geq N \geq N_{x_j}$, we have that $f_n(y)$ is close to $f(y)$, and in fact $|f_n(y) - f(y)| < \varepsilon$.

Therefore, f_n converges uniformly to f on $[a, b]$.

3.

PROOF (Extreme Value Theorem)

A function continuous on a closed bounded interval $[a, b]$ attains its maximum and minimum values.

Let f be continuous on $[a, b]$.

Let $B = \{ f(x) : x \in [a, b] \}$.

Then B is non-empty (since $f(a) \in B$). Also, B is bounded above because every continuous function on a closed, bounded interval is bounded.

Let $M = \sup B$ (least upper bound of B).

Assume that for all $x \in [a, b]$, $f(x) \neq M$.

Then $M - f(x) > 0$ for all $x \in [a, b]$.

Define the function:
 $g(x) = 1 / (M - f(x))$.

Then g is continuous on $[a, b]$.

Since continuous functions on closed bounded intervals are bounded, there exists $K > 0$ such that:
 $g(x) \leq K$ for all $x \in [a, b]$.

So:

$$1 / (M - f(x)) \leq K$$

which implies:
 $M - f(x) \geq 1 / K$

so:

$$f(x) \leq M - 1 / K \text{ for all } x \in [a, b].$$

Thus $M - 1 / K$ is an upper bound of B , and $M - 1 / K < M$.

This contradicts that M is the least upper bound.

Therefore, there exists $x \in [a, b]$ such that $f(x) = M$.

Now apply the same argument to $-f(x)$. Then $-f$ attains a maximum, which means f attains a minimum.

Therefore, f attains both a maximum and a minimum on $[a, b]$

LITERARY TRANSLATION (an excerpt from the story: *Professor and Colonel* by RUTH BERMAN)

They wandered a while in silence. Robert was not an entirely comfortable person for companionship in a formal garden. If you pointed out the beautiful sunflowers and then went on to the hollyhocks, you might go quite a distance before you realized that your companion was still transfixed by a sunflower, counting the florets in each spiral and charting the pattern of clockwise and widdershins spirals, apparently blind to the gold color and the beautiful halo made by the petals, which he called the ligulate flowers.

"Yes, it's symmetrical," he agreed enthusiastically, when James tried to call his attention to the petals, or ligulate flowers, as the case might be. James gave it up and asked if Robert had gone any further with his sequel to *The Dynamics of an Asteroid*.

Robert let the sunflower go and looked sidelong at his brother. "No."

"Oh," said James, feeling startled. He knew Robert had been delighted with his success at finding an equation for the problem of several bodies. James tried to think of something sensible yet consoling to say about not continuing. "Perhaps something with more practical applications would be better? Knowing where an asteroid will be doesn't shake the Earth, I suppose?"

"That isn't the point!" said Robert. Then he shrugged. "But no one else seemed to understand, either."

"What is the point?" said James.

"The point is precision. If you can calculate a path, you can even tell where atoms of matter, or light, or energy will be at any point in time. Not that an atom's orbit is an asteroid's. But if we could calculate the relationship between energy and matter, we could fuel engines to —" He stopped and laughed at James's confusion. "Never mind. But it's all a question of trajectories. Asteroids move, atoms move, we move. Perhaps if we knew enough," he added thoughtfully, "we could turn fortune telling into truth and trace the entire past and future of any object sufficiently well defined."

them, so that to glance in any direction was to know at once where one stood and how the surrounding landscape incorporated that point. Do you know what they did, André Le Nôtre, and those other old seventeenth-century gardeners who planted the gardens of the Sun King?"

"Laid out Versailles," James suggested.

"They integrated space," Robert said, not bothering to acknowledge the simpler answer. "Look how those straight diagonals tell you where you are and where you can go. Look how the reflecting pools double the space, and define it, in those long straight edges. It's no wonder Descartes discovered at the same time how to define space numerically. The old medieval gardens, the kinds the monasteries cultivated, were only arithmetical, adding square to square, expanding in squares as far as their walls would let them, without any sight or sense of the whole. Here the paths give measure to the shape. What André felt is what old René learned to analyze."

"I like an English garden better," said James. "It's more natural."