The translation of a technical text.

PROOF (Heine-Borel Theorem)

Let $a < b$ be two real numbers, and let T be an open cover of [a,b]. Then T contains a finite subcover of [a,b].

• Let $a < b$ be two real numbers, and let T be an open cover of [a,b].

• Define the set

S = { x in [a,b] | T has a finite subcover that covers the interval [a,x] }.

• The set T is an open cover of [a,b], and a belongs to [a,b], so T must contain at least one open interval (p,q) which contains the point a, that is,

$p < a < q$.

Since the interval [a,a] is covered by (p,q) in T, the point a belongs to S. Hence S is nonempty.

• The set S is bounded above by b.

• Since S is nonempty and bounded above, it has a least upper bound r.

• Since r is at least a and at most b, we have r in [a,b]. Therefore there is an interval (p,q) in T containing r, so

$p < r < q$.

• Since $p < r$ and r is the least upper bound of S, p cannot be an upper bound of S. Therefore there exists y in S such that

$p < y$.

This means there is a finite set of intervals in T covering [a,y].

• Let

z = min((r+q)/2, b).

Since z >= r and z belongs to (p,q), adding the interval (p,q) to the finite family covering [a,y] gives a finite family of intervals in T covering [a,z]. Thus z belongs to S.

• But r is the least upper bound of S, so z <= r.

Because

(r+q)/2 > r,

it follows that the only way z = min((r+q)/2, b) can satisfy z <= r is if

z = b.

• Since z belongs to S, we get b belongs to S.

Therefore T has a finite subcover of [a,b], completing the proof.

PROOF (Bolzano-Weierstrass Theorem)

Every infinite bounded set of real numbers has an accumulation point.

• Let A be an infinite bounded set of real numbers.

• Because A is bounded, it has a lower bound a and an upper bound b, so

A is contained in [a,b].

• Define the set

S = { x >= a | [a,x] intersect A is finite }.

• Note that a belongs to S, since [a,a] intersect A is finite. Therefore S is nonempty.

• If z >= b, then

[a,z] intersect A = A,

which is infinite. Hence z does not belong to S. Therefore S is bounded above by b.

• By the Completeness Axiom, S has a least upper bound p.

• Let epsilon > 0.

Since p - epsilon < p, the number p - epsilon is not an upper bound of S.

Therefore there exists y in S such that

y > p - epsilon.

It follows that there are only finitely many elements of A less than or equal to y.

• Also,

p + epsilon > p,

so p + epsilon does not belong to S.

Therefore

[a, p + epsilon] intersect A

is infinite.

• Since only finitely many elements of A are less than or equal to y, but infinitely many elements of A lie in [a, p + epsilon], there must be infinitely many elements of A between

p - epsilon and p + epsilon.

• Hence there is an element of A, different from p, lying within epsilon of p.

• Since this is true for every epsilon > 0, p is an accumulation point of A.

This completes the proof.

PROOF (Intermediate Value Theorem)

Let f be continuous on the interval [a,b], and let c and d belong to [a,b]. If y is any value between f(c) and f(d), then there exists x between c and d such that

f(x) = y.

• Let f be a function continuous on [a,b], and let c and d be points in [a,b].

• Let y be any value between f(c) and f(d).

• Without loss of generality, assume that

c <= d

and

f(c) <= y <= f(d).

• Define the set

S = { x in [c,d] | f(x) <= y }.

• The set S is nonempty because f(c) <= y, so c belongs to S.

• The set S is bounded above by d.

• By the Completeness Axiom, S has a least upper bound s. Since c <= s <= d, the point s lies in [a,b].

• Suppose that

f(s) < y.

By continuity of f at s, there exists delta > 0 such that whenever x belongs to [a,b] and

|x - s| < delta,

we have

|f(x) - f(s)| < (y - f(s))/2.

In particular,

f(x) < y.

So there exists some x > s with f(x) < y, which means x belongs to S.

This contradicts the fact that s is an upper bound of S.

• Suppose that

f(s) > y.

By continuity of f at s, there exists delta > 0 such that whenever x belongs to [a,b] and

$|x - s| < delta$,

we have

$|f(x) - f(s)| < (f(s) - y)/2$.

In particular,

$f(x) > y$.

Therefore for every x in the interval (s - delta, s),

$f(x) > y$.

So s - delta is an upper bound of S.

This contradicts the fact that s is the least upper bound of S.

• Since both f(s) < y and f(s) > y are impossible, it follows that

$f(s) = y$.

Thus there exists x = s between c and d such that

$f(x) = y$.

A literary translation: an excerpt from Rudy Bucker's story (Golden age)

This completes the proof.

Then they switched on the second tape

.

Watson was the only one of us who had truly mastered the Kunen paper on which this tape was based. But he had refused to have his brain patterns recorded. Instead, he had constructed the whole thing as an artificial design in our parameter space.

The tape played in my head without words or pictures.

There was a measurable cardinal.

Suddenly I knew its properties in the same silent way that I knew my own body.

I did something to the cardinal, and it transformed itself, changing the concepts clustered around it.

This happened again and again.

With a feeling of light-headedness, I felt myself moving outside this endless self-transformation, comprehending it from the outside.

I selected a certain subconstellation of the whole process and wrapped it in its logical hull.

Suddenly I understood a theorem I had always wondered about.

When the tape ended, I begged my colleagues for an hour of privacy.

I had to think about iterated ultrapowers some more.

I rushed to the library and took out Kunen's paper.

But the lucidity was gone.

I began to stumble over the notation, the subscripts and superscripts.

I was baffled by the incomplete proofs.

I kept forgetting the definitions.

Already the actual content of the main theorem escaped me.

I realized then that the Moddler was a success.

You could enjoy mathematics, even mathematics you could not normally understand.

We all got a little drunk that night.

Somewhere around midnight I found myself walking along the edge of the woods with Mies.

He was humming softly, marking time with gentle nods of his head.

We stopped while I lit my thirtieth cigarette of the day.

In the flare of the match, I thought I noticed something strange in Mies's expression.

“What is it?” I asked, exhaling smoke.

“The music...” he began.

“The music most people listen to is not good.”

I did not understand what he meant, and began my usual defense of rock music.

“Muzak,” Mies interrupted.

“Isn't that what you call it... what they play in airports?”

“Yeah. Easy listening.”

“Do you really expect that the official taste in mathematics will be any better?

If everyone were to sit under the Moddler, what kind of mathematics would they ask for?”

I recoiled from the suggestion.

“Don't worry, Mies.

There are objective standards of mathematical truth.

No one will undermine them.

We're headed for a new golden age.”