Homework 14.4.2026.

Convergent Real Sequence Has Unique Limit

Theorem

Let (s_n) be a real sequence.

Then (s_n) can have at most one limit.

Proof 1

We prove by contradiction.

Suppose that the sequence (s_n) converges to l and also converges to m.

That is:

$\lim_{n\to\infty} s_n = l$

and

$\lim_{n\to\infty} s_n = m$

Assume $l \neq m$.

Define:

$\epsilon = |l - m| / 2$

Since $l \neq m$, we have $\epsilon > 0$.

Because $s_n \to l$, by the definition of convergence:

There exists $N1 \in N$ such that for every $n > N1$,

$|s_n - l| < \epsilon$

Because $s_n \to m$:

There exists $N2 \in N$ such that for every $n > N2$,

$|s_n - m| < \epsilon$

Now let:

$N = \max\{N1, N2\}$

Then N is greater than both N1 and N2, so both inequalities hold for n = N.

Hence:

$|s_N - l| < \text{epsilon}$

and

$|s_N - m| < \text{epsilon}$

Now write:

$|l - m|$

$|l - s_N + s_N - m|$

Using the triangle inequality:

$|l - m| \leq |l - s_N| + |s_N - m|$

But each term on the right is less than epsilon, so:

$|l - m| < \text{epsilon} + \text{epsilon} = 2\ \text{epsilon}$

Since epsilon = $|l - m| / 2$, we get:

$|l - m| < |l - m|$

This is impossible.

Contradiction.

Therefore our assumption $l \neq m$ was false.

Hence:

$l = m$

So a convergent real sequence has only one limit.

QED

Short Intuition

If a sequence converged to two different numbers, then eventually all terms would have to be very close to both numbers at the same time.

But if the two numbers are different, this cannot happen forever.

Theorem: Order Preserved in the Limit (Sandwich Inequality Form)

Let (x_n), (a_n), and (b_n) be convergent real sequences.

Suppose:

$x_n \to x$
$a_n \to a$
$b_n \to b$

Assume that there exists $N \in \mathbb{N}$ such that for every $n \geq N$:

$a_n \leq x_n \leq b_n$

Then:

$a \leq x \leq b$

Proof

We first prove:

$x \geq a$

We use contradiction.

Assume:

$x < a$

Choose:

$\text{epsilon} = (a - x) / 2$

Since $x < a$, we have epsilon > 0.

Because $x_n \to x$, there exists $M1 \in \mathbb{N}$ such that for all $n \geq M1$:

$|x_n - x| < \text{epsilon}$

So:

$x - \text{epsilon} < x_n < x + \text{epsilon}$

Substitute $\text{epsilon} = (a - x)/2$:

$x_n < x + (a - x)/2 = (x + a)/2$

Thus:

for all $n \geq M1$,

$x_n < (x + a)/2$

Because $a_n \to a$, there exists $M2 \in \mathbb{N}$ such that for all $n \geq M2$:

$|a_n - a| < \text{epsilon}$

So:

$a - \text{epsilon} < a_n < a + \text{epsilon}$

Hence:

$a_n > a - (a - x)/2 = (x + a)/2$

Thus:

for all $n \geq M2$,

$a_n > (x + a)/2$

Now let:

$M = \max\{N, M1, M2\}$

Then for every $n \geq M$, all three facts hold:

1. $a_n \leq x_n$ (given hypothesis)
2. $x_n < (x + a)/2$
3. $a_n > (x + a)/2$

From (2) and (3):

$x_n < (x + a)/2 < a_n$

So:

$x_n < a_n$

But this contradicts:

$a_n \leq x_n$

Contradiction.

Therefore the assumption $x < a$ is false.

Hence:

$x \geq a$

Now we prove:

$x \leq b$

The argument is completely analogous.

Assume $x > b$.

Choose:

epsilon = $(x - b)/2$

Using convergence of x_n and b_n, for sufficiently large n we obtain:

$x_n > (x + b)/2 > b_n$

But the hypothesis says:

$x_n \leq b_n$

Contradiction.

Therefore:

$x \leq b$

Combining both results:

$a \leq x \leq b$

Conclusion

If eventually every term of x_n lies between a_n and b_n, then the limit of x_n must lie between the limits of a_n and b_n.

Theorem

Limit of Subsequence Equals Limit of Sequence

Let (x_n) be a sequence in a topological space.

Suppose:

$x_n \to l$

Let (x_{n_r}) be any subsequence of (x_n).

Then:

$x_{n_r} \to l$

In words:

Every subsequence of a convergent sequence converges to the same limit.

Special Case: Real Numbers

Let (x_n) be a real sequence.

If:

$\lim_{n\to\infty} x_n = l$

and (x_{n_r}) is a subsequence of (x_n),

then:

$\lim_{r\to\infty} x_{n_r} = l$

Proof

We use the definition of convergence.

Assume:

$x_n \to l$

This means:

For every epsilon > 0, there exists $N \in \mathbb{N}$ such that for all $n \geq N$:

$|x_n - l| < \epsilon$

Now let (x_{n_r}) be a subsequence.

By definition of subsequence:

$n_1 < n_2 < n_3 < \ldots$

So the indices are strictly increasing.

Hence:

$n_r \geq r$

and in particular:

$r \geq N$ implies $n_r \geq r \geq N$

Therefore, whenever $r \geq N$:

$|x_{n_r} - l| < \epsilon$

because n_r is one of the original indices larger than or equal to N.

So we have shown:

For every epsilon > 0, there exists N such that for all $r \geq N$:

$|x_{n_r} - l| < \epsilon$

This is exactly the definition that:

$x_{n_r} \to l$

Conclusion

Every subsequence of a convergent sequence has the same limit as the original sequence.

Intuition

If all late terms of the original sequence are close to l, then any subsequence, which only selects some of those late terms, must also stay close to l.

Intersection of Two Subspaces Is a Subspace

Problem

Let H and K be subspaces of a vector space V.

Define:

$H \cap K = \{ v \in V \mid v \in H \text{ and } v \in K \}$

Show that $H \cap K$ is a subspace of V.

Proof

To prove that $H \cap K$ is a subspace, we check the three subspace conditions:

1. Nonempty (contains zero vector)
2. Closed under addition
3. Closed under scalar multiplication

Step 1: Zero vector belongs to $H \cap K$

Because H and K are subspaces, each contains the zero vector.

So:

$0 \in H$
$0 \in K$

Therefore:

$0 \in H \cap K$

Hence $H \cap K$ is nonempty.

Step 2: Closed under vector addition

Take any vectors:

$u, v \in H \cap K$

By definition of intersection, this means:

$u \in H$ and $u \in K$
$v \in H$ and $v \in K$

Since H is a subspace, it is closed under addition:

$u + v \in H$

Since K is also a subspace, it is closed under addition:

$u + v \in K$

Therefore:

$u + v \in H$ and $u + v \in K$

So:

$u + v \in H \cap K$

Hence $H \cap K$ is closed under addition.

Step 3: Closed under scalar multiplication

Take any scalar c and any vector $v \in H \cap K$.

Then:

$v \in H$ and $v \in K$

Because H is a subspace:

$cv \in H$

Because K is a subspace:

$cv \in K$

Therefore:

$cv \in H \cap K$

So $H \cap K$ is closed under scalar multiplication.

Conclusion

$H \cap K$ satisfies all three conditions for a subspace.

Therefore:

$H \cap K$ is a subspace of V.

To prove closure under addition:

If $u, v \in H \cap K$, then both vectors lie in H and in K.

Since both H and K are subspaces, each is closed under addition, so:

$u + v \in H$
$u + v \in K$

Therefore:

$u + v \in H \cap K$

Theorem

Let V be a vector space over a division ring K.

Let $S \subseteq V$ be a subset of V.

Then the linear span $\operatorname{span}(S)$ is a subspace of V.

Proof 1

First, suppose that $S = \emptyset$.

By definition of linear combination of the empty set, it follows that:

$\operatorname{span}(\emptyset) = \{0\}$

where 0 denotes the zero vector of V.

Since $\{0\}$ is the trivial subspace, it follows that:

$\{0\}$ is a subspace of V.

Now suppose that S is non-empty.

Every vector $v \in \operatorname{span}(S)$ has the form:

$v = \sum(k=1 \text{ to } n) \lambda_k v_k$

where $n \in N$, $\lambda_k \in K$, and $v_k \in S$.

We use the Two-Step Subspace Test.

Step 1: Closure under scalar multiplication

Let $\lambda \in K$ and $v \in \operatorname{span}(S)$.

Then:

$\lambda v = \lambda(\sum(k=1 \text{ to } n) \lambda_k v_k)$

Using distributivity:

$\lambda v = \sum(k=1 \text{ to } n) (\lambda \lambda_k) v_k$

Each term is still a linear combination of elements of S, hence:

$\lambda v \in \operatorname{span}(S)$

Step 2: Closure under addition

Let $v, w \in \mathrm{span}(S)$.

Then:

$v = \sum_{k=1}^{n} \lambda_k v_k$
$w = \sum_{l=1}^{m} \mu_l w_l$

where all v_k and w_l belong to S.

Then:

$v + w = \sum_{k=1}^{n} \lambda_k v_k + \sum_{l=1}^{m} \mu_l w_l$

This is again a finite linear combination of vectors from S, hence:

$v + w \in \mathrm{span}(S)$

Since span(S) is closed under addition and scalar multiplication, it follows that:

span(S) is a subspace of V.

List Longer Than the Dimension Is Linearly Dependent

Theorem

Let V be a finite-dimensional vector space with:

$\dim(V) = n$

Then any list of vectors in V containing more than n vectors is linearly dependent.

In other words:

If $(v_1, v_2, ..., v_m)$ is a list in V and $m > n$, then the list is linearly dependent.

Proof (standard argument via dimension)

Let $\dim(V) = n$.

Then there exists a basis:

$B = (b_1, b_2, ..., b_n)$

So B spans V and is linearly independent.

Step 1: Use the spanning property

Since B spans V, every vector in V can be written as a linear combination of b1, ..., bn.

In particular, each vi (for i = 1, ..., m) can be expressed as:

vi = a1i b1 + a2i b2 + ... + ani bn

So all m vectors lie in the span of n basis vectors.

Step 2: Apply the key idea (pigeonhole principle for vectors)

We now consider the list:

(v1, v2, ..., vm)

expressed in terms of n basis vectors.

Since there are more vectors (m > n) than basis directions (n), some vector must be expressible as a linear combination of the previous ones.

More formally, we build a matrix of coefficients of size n × m.

This matrix has more columns than rows, so its columns are linearly dependent.

Step 3: Conclude linear dependence

Thus there exist scalars:

c1, c2, ..., cm not all zero

such that:

c1v1 + c2v2 + ... + cmvm = 0

Therefore the list is linearly dependent.

Conclusion

Any list of more than n vectors in an n-dimensional vector space is linearly dependent.

2.

Translate this story into Serbian.

Peter’s a Real Flake (an excerpt from Greg Bear’s story ‘’Tangent’’)

“Peter’s a real flake. He’s trying to see certain directions.”
“I know,” Lauren said, sighing.

“I’m going home now,” Pal said. “I’ll be back, though… if it’s all right with you. Peter invited me.”
“I’m sure that it will be fine,” Lauren replied dubiously.

“He’s going to let me learn the Tronclavier.”
With that, Pal smiled radiantly and exited through the kitchen door.

When she retrieved the tray, she found Peter leaning back in his chair, eyes closed. The figures on the screen patiently folded and unfolded, cubes continuously passing through one another.

“What about Hockrum’s work?” she asked.
“I’m on it,” Peter replied, eyes still closed.

Lauren called Pal’s adoptive mother on the second day to report his location. The woman said it was fine.

“Sometimes he’s a little pest. Send him home if he causes trouble—but not right away! Give me a rest,” she said, then laughed nervously.

Lauren thanked her and hung up.

Peter and the boy sat in the kitchen, filling paper with line drawings.

“Peter’s teaching me how to use his program,” Pal said.

“Did you know,” Tuthy said in his most professorial tone, “that a cube, intersecting a flat plane, can produce a number of geometrically different cross sections?”

Pal looked at the sketch.

“Sure,” he said.

“If a cube is pushed through a plane, it can appear to a two-dimensional creature living there—let’s call him a Flatlander—to be a triangle, rectangle, trapezoid, rhombus, or square. As it moves through, these shapes grow, change, and disappear.”

“Sure,” Pal said, tapping his sneakered toe. “It’s easy. Like in that book you showed me.”

“And a sphere pushed through a plane would first appear as a point, then as a circle, which grows and shrinks back to a point again.”

He sketched the figures, staring in awe at the idea of higher-dimensional intrusion.