

## Uniqueness Theorem for Vector Representation in a Basis

**Theorem 1:** Every vector in a vector space is represented by a vector basis  $B$  through a unique linear combination of scalars.

Definition:

Let  $B = \{v_1, v_2, \dots, v_n\}$  be a basis of the vector space  $V$  over the field  $K$ . Then every vector  $v$  in  $V$  can be expressed as a unique linear combination:

$$v = a_1*v_1 + a_2*v_2 + \dots + a_n*v_n$$

with coefficients  $a_1, \dots, a_n$  in  $K$ .

Proof:

Consider a generic vector  $v$  in the vector space  $V$ :

$v$  in  $V$

The basis  $B$  consists of  $n$  vectors:  $\{v_1, v_2, \dots, v_n\}$

Hence, the vector  $v$  can be written as a linear combination of the  $n$  basis vectors:

$$v = a_1*v_1 + a_2*v_2 + \dots + a_n*v_n$$

Assume, for the sake of contradiction, that the vector can also be written as another linear combination:

$$v = b_1*v_1 + b_2*v_2 + \dots + b_n*v_n$$

Since it is the same vector, the two linear combinations must be equal:

$$a_1*v_1 + a_2*v_2 + \dots + a_n*v_n = b_1*v_1 + b_2*v_2 + \dots + b_n*v_n$$

Move all terms to one side:

$$(a_1 - b_1)*v_1 + (a_2 - b_2)*v_2 + \dots + (a_n - b_n)*v_n = 0$$

Since  $\{v_1, \dots, v_n\}$  is a basis, the vectors are linearly independent.

A linear combination of independent vectors equals zero only if all coefficients are zero:

$$a_1 - b_1 = 0$$

$$a_2 - b_2 = 0$$

...

$$a_n - b_n = 0$$

Therefore, the coefficients are equal:

$$a_1 = b_1$$

$$a_2 = b_2$$

...

$$a_n = b_n$$

This proves that the linear combination representing a vector through a basis of the vector space is unique.

Alternate Proof:

A basis is a minimal set of vector generators.

Any vector in the space can be represented by a linear combination of the  $n$  vectors of the basis:

$$v = a_1*v_1 + \dots + a_n*v_n$$

Suppose two distinct linear combinations represent the same vector:

$$v = a_1*v_1 + \dots + a_n*v_n$$

$$v = b_1*w_1 + \dots + b_m*w_m$$

Subtract to get zero:

$$a_1*v_1 + \dots + a_n*v_n - b_1*w_1 - \dots - b_m*w_m = 0$$

Let  $k = \min(n, m)$ .

Assume that up to the  $k$ -th element, the vectors are the same, and after that they differ:

$$v_1 = w_1$$

...

$$v_k = w_k$$

$$v_{k+1} \neq w_{k+1}$$

...

Then the equation can be rewritten:

$$v_1*(a_1 - b_1) + \dots + v_k*(a_k - b_k) + (a_{k+1}*v_{k+1} - b_{k+1}*w_{k+1}) + \dots = 0$$

The equation equals zero only if all corresponding coefficients are equal:

$$a_1 = b_1$$

...  
 $a_k = b_k$   
 $a_{k+1} = 0$   
...  
 $a_n = 0$   
 $b_{k+1} = 0$   
...  
 $b_m = 0$

Hence, up to the  $k$ -th element, the coefficients match, and beyond that, the remaining coefficients are zero.

This confirms the uniqueness of the representation.

**Theorem 2:** Every finite-dimensional subspace of a normed linear space is closed.

This is proved by showing that it is complete with respect to the induced norm (since all norms on a finite-dimensional space are equivalent) and any complete subspace of a normed space is closed.

Proof:

1. Define the Subspace:

Let  $X$  be a normed linear space and  $Y$  be a finite-dimensional subspace of  $X$ .

2. Choose a Basis:

Let  $\{v_1, v_2, \dots, v_n\}$  be a basis for  $Y$ . Any vector  $y$  in  $Y$  can be uniquely written as a linear combination:

$$y = a_1 v_1 + a_2 v_2 + \dots + a_n v_n$$

where  $a_1, \dots, a_n$  are scalars.

3. Define an Equivalent Norm:

Define a new norm on  $Y$  based on the coefficients:

$$\|y\|_{\text{new}} = \max(|a_1|, |a_2|, \dots, |a_n|)$$

4. Equivalence of Norms:

It is a standard result that all norms on a finite-dimensional space are equivalent.

Therefore, the original norm  $\|\cdot\|$  restricted to  $Y$  is equivalent to the norm  $\|\cdot\|_{\text{new}}$ .

5. Completeness:

The space  $Y$  is complete with respect to  $\|\cdot\|_{\text{new}}$  (since finite-dimensional spaces

are complete). Because the norms are equivalent,  $Y$  is also complete with respect to the original norm  $\| \cdot \|$ .

6. Closedness:

Every complete subspace of a normed linear space is closed. Therefore,  $Y$  is closed in  $X$ .

Literary – technical translation

### **NORMAN KAGAN (page 9)**

#### Four Brands of Impossible

Harry Mandel hailed me from the psychology library. I grinned and waited for him to join me. Harry is a swell guy, and besides he's a psychology major, not a math competitor. He joined me, puffing, a moment later. "Summer job hunting?" "Yeah. I've got a couple of interviews arranged: Serendipity, Inc., and the Virgin Research Corporation." "Me, too," said Harry. He gestured at my tousled hair. "Getting ready? Physical appearance is very important, you know." The short, pudgy psychology major pumped his legs to keep up with me. It was a warm, comfortable day on the Multiversity campus. The long rows of wooden chairs were already set up for graduation, and here and there was a girl in long hair and Levis, or a bearded boy with a guitar. Early summer-session people. Lazy jerks.

"What's the word on the companies?"

Mandel wrinkled up his forehead. "Personnel men—not technical people. So if you've got the grades, go all out—anything to avoid the paper barrier."

"Any specific suggestions?"

"Mmmm—well. Fester pulled a full-scale epileptic fit—but then he's nearly a five point. If you're just bright, a few eccentricities ought to do it. I'm trying my bug-on-the-walls gambit."

"You mean the one where you pretend there's a bug that crawls all over the walls behind the interviewer, and you follow it with

your eyes." "No—that was last year. In this one I sort of scrunch up in the chair, cowering—give 'em the impression I can't stand confined spaces, need lots of room—like, say. New Mexico or Arizona. I'm sick of this East Coast weather, and the Virgin Research Corporation has labs in New Mexico."

"That's for me, too. I'll see what I can think up." In the 1980s it's practically impossible to get a summer job in the sciences—not that the big science and engineering corporations don't want you. They do. But to apply, you've got to submit about a ton of paper work—eight commendations, four transcripts, character references, handwriting samples, personality profiles, certificates and forms and diplomas. Who has the energy?