Assignment 1: . Please translate the following text from English to Serbian (Terence Tao 155-161)

2.6 SUPREMUM OF SETS OF EXTENDED REALS

Definition 6.2.6 (Supremum of sets of extended reals)

Let E be a subset of R*.

The supremum sup(E), also called the least upper bound of E, is defined as follows:

(a) If E is contained in R (that is, neither +infinity nor -infinity belongs to E), then sup(E) is defined as in Definition 5.5.10.

(b) If E contains +infinity, then

sup(E) = +infinity.

(c) If E does not contain +infinity but does contain -infinity, then

sup(E) = sup(E - {-infinity})

which falls under case (a).

INFIMUM

The infimum inf(E), also called the greatest lower bound of E, is defined by

inf(E) = -sup(-E)

where

-E = { -x : x belongs to E }.

EXAMPLE 6.2.7

Let

E = {-1, -2, -3, -4, ... } U {-infinity}.

Then

sup(E)
= sup(E - {-infinity})
= -1.

Also,

inf(E)
= -sup(-E)
= -(+infinity)
= -infinity.

EXAMPLE 6.2.8

The set

{0.9, 0.99, 0.999, 0.9999, ...}

has

infimum = 0.9

supremum = 1.

Note:

The supremum does not belong to the set, but it is "touching" the set from the right.

EXAMPLE 6.2.9

The set

{1, 2, 3, 4, 5, ...}

has

infimum = 1

supremum = +infinity.

EXAMPLE 6.2.10

Let E be the empty set.

Then

sup(E) = -infinity

inf(E) = +infinity.

This is the only case where

sup(E) < inf(E).

INTUITIVE PICTURE

Think of the real line with

+infinity far to the right,

-infinity far to the left.

Imagine a piston starting at +infinity and moving left.

It stops when it reaches the set E.

The stopping point is sup(E).

Similarly, imagine a piston starting at -infinity and moving right.

It stops when it reaches E.

The stopping point is inf(E).

If E is empty, the pistons pass through each other:

sup(E) = -infinity

inf(E) = +infinity.

THEOREM 6.2.11

Let E be a subset of R*.

Then:

(a) For every x in E,

x <= sup(E)

and

x >= inf(E).

(b) If M is an upper bound for E, meaning

x <= M for all x in E,

then

sup(E) <= M.

(c) If M is a lower bound for E, meaning

x >= M for all x in E,

then

inf(E) >= M.

Proof: See Exercise 6.2.2.

EXERCISES

Exercise 6.2.1

Prove Proposition 6.2.5.

Hint:
You may need Proposition 5.4.7.

Exercise 6.2.2

Prove Theorem 6.2.11.

Exercise 6.2.2

Prove Proposition 6.2.11.

Hint:
You may need to break into cases depending on whether $+\infty$ or $-\infty$ belongs to E.
You may use Definition 5.5.10 when E consists only of real numbers.

6.3 Suprema and Infima of Sequences

Definition 6.3.1 (Supremum and Infimum of a Sequence)

Let (a_n)_(n>=m) be a sequence of real numbers.

Define:

sup(a_n)_(n>=m)

to be the supremum of the set

{ a_n : n >= m }

and define

inf(a_n)_(n>=m)

to be the infimum of the same set

{ a_n : n >= m }.

Remark 6.3.2

The quantities

sup(a_n)_(n>=m)

and

inf(a_n)_(n>=m)

are sometimes written as

sup_(n>=m) a_n

and

inf_(n>=m) a_n

respectively.

Example 6.3.3

Let

a_n = (-1)^n

Then the sequence is

-1, 1, -1, 1, ...

The set

{ a_n : n >= 1 }

is simply

{ -1, 1 }.

Therefore

$\sup(a_n)_{(n>=1)} = 1$

and

$\inf(a_n)_{(n>=1)} = -1$.

Example 6.3.4

Let

$a_n = 1/n$

Then the sequence is

$1, 1/2, 1/3, 1/4, ...$

The set

$\{ a_n : n >= 1 \}$

is

$\{ 1, 1/2, 1/3, 1/4, ... \}$.

Therefore

$\sup(a_n)_{(n>=1)} = 1$

$\inf(a_n)_{(n>=1)} = 0$

(Exercise 6.3.1).

Notice that the infimum is not actually a member of the sequence, although the sequence gets arbitrarily close to it.

Thus it is not always accurate to think of the supremum and infimum as the largest and smallest terms of the sequence.

Example 6.3.5

Let

$a_n = n$

Then the sequence is

$1, 2, 3, 4, \ldots$

The set

$\{ a_n : n >= 1 \}$

is

$\{ 1, 2, 3, 4, \ldots \}$.

Therefore

$\sup(a_n)_{(n>=1)} = +\infty$

$\inf(a_n)_{(n>=1)} = 1$

Bounded Sequences

The previous example shows that a supremum or infimum may be $+\infty$ or $-\infty$.

However, suppose the sequence $(a_n)_{(n>=m)}$ is bounded by M.

Then every term satisfies

$-M <= a_n <= M$.

Hence the set

$\{ a_n : n >= m \}$

has

M

as an upper bound and

$-M$

as a lower bound.

Since the set is nonempty, its supremum and infimum must be real numbers (not $+\infty$ or $-\infty$).

Proposition 6.3.6
(Least Upper Bound Property)

Let $(a_n)_{(n>=m)}$ be a sequence of real numbers.

Let

$x = \sup(a_n)_{(n>=m)}$.

Then:

(1) For every $n >= m$,

$a_n <= x$.

(2) If M is an upper bound for the sequence, meaning

$a_n <= M$

for all $n >= m$,

then

$x <= M$.

(3) For every extended real number y satisfying

$y < x$,

there exists at least one $n >= m$ such that

$y < a_n <= x$.

Proof:

See Exercise 6.3.2.

Remark 6.3.7

There is a corresponding proposition for infima.

To obtain it, reverse all order relations:

- upper bounds become lower bounds,
- <= becomes >=,
- supremum becomes infimum.

The proof is exactly the same.

Application

In the previous section we proved that every convergent sequence
is bounded.

A natural question is whether the converse is true.

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re All Bounded Sequences Convergent?

The answer is NO.

Example:

1, -1, 1, -1, ...

This sequence is bounded, but it is not Cauchy,
therefore it is not convergent.

However, if a sequence is both:

(1) bounded, and
(2) monotone (increasing or decreasing),

then it must converge.

Proposition 6.3.8
(Monotone Bounded Sequences Converge)

Let $(a_n)_{(n>=m)}$ be a sequence of real numbers.

Assume:

(1) The sequence has a finite upper bound $M \in R$.

(2) The sequence is increasing:

$a_{(n+1)} >= a_n$

for all $n >= m$.

Then the sequence converges, and

$\lim_{(n \to \infty)} a_n$
=
$\sup(a_n)_{(n>=m)}$
$<= M$

Proof:
See Exercise 6.3.3.

Similarly:

If a sequence is

(1) bounded below, and
(2) decreasing,

then it converges and

$\lim_{(n \to \infty)} a_n$
=
$\inf(a_n)_{(n>=m)}$

Definition

A sequence is called monotone if it is either

- increasing, or
- decreasing.

Therefore:

A monotone sequence converges
if and only if
it is bounded.

(Proposition 6.3.8 + Corollary 6.1.17)

Example 6.3.9

Sequence:

3,
3.1,
3.14,
3.141,
3.1415,
...

This sequence is increasing.

It is bounded above by 4.

Therefore, by Proposition 6.3.8,
it converges to some real number less than or equal to 4.

Proposition 6.3.8 guarantees that a limit exists.

However, it does not directly tell us what the limit is.

The next proposition shows how to determine a limit.

Proposition 6.3.10

Let

$0 < x < 1$

Then

$\lim_{n \to \infty} x^n = 0$

Proof

Since

$0 < x < 1$

the sequence

(x^n)

is decreasing.

Why?

Because multiplying by x makes each term smaller.

Also,

0

is a lower bound of the sequence.

Therefore, by Proposition 6.3.8
(using infima instead of suprema),

(x^n)

converges to some limit L.

Since

$x^{(n+1)} = x \cdot x^n$

the limit laws imply that

$(x^{(n+1)})$

converges to

xL.

But

$(x^{(n+1)})$

is simply the sequence

(x^n)

shifted by one position.

A shift does not change the limit.

Therefore

$xL = L$

which gives

$L(x - 1) = 0$

Since

$x \neq 1$

we conclude

$L = 0$.

Hence

$\lim_{n \to \infty} x^n = 0$.

□

Note:

This proof does NOT work when

$x > 1$.

(See Exercise 6.3.4.)

Exercises

Exercise 6.3.1

Verify the claim in Example 6.3.4.

Exercise 6.3.2

Prove Proposition 6.3.6.

Hint:
Use Theorem 6.2.11.

Exercise 6.3.3

Prove Proposition 6.3.8.

Hint:

Use Proposition 6.3.6 together with the fact that

a_n

is increasing.

Show that

a_n → sup(a_n)_(n>=m).

Exercise 6.3.4

Explain why Proposition 6.3.10 fails when

x > 1.

Show that

(x^n)

diverges whenever

x > 1.

Hint:

Use proof by contradiction.

Use the identity

(1/x)^n · x^n = 1

and the limit laws from Theorem 6.1.19.

Compare with Example 1.2.3 and identify the flaw in the reasoning there.

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6.4 Limsup, Liminf, and Limit Points

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Consider the sequence

1.1,
-1.01,
1.001,
-1.0001,
1.00001,
...

Informally, this sequence does not converge.

Half the terms approach

1

and half the terms approach

-1.

The sequence therefore converges neither to 1 nor to -1.

For example:

It never becomes permanently within 1/2 of 1.

It never becomes permanently within 1/2 of -1.

Nevertheless, the numbers

1

and

-1

seem to be values that the sequence repeatedly approaches.

To make this idea precise we introduce the notion of a limit point.

Assignmenmt 2: ry Niven — "Convergent Series"

"You'd be immortal for what's left of your twenty-four hours."

He grinned. His teeth were coal black.

"Better hurry. Time's running out."

Time, I thought.

Okay.

All or nothing.

"Here's my wish. Stop time from passing outside of me."

"Easy enough. Look at your watch."

I didn't want to take my eyes off him, but he just exposed his black teeth again.

So—I looked down.

There was a red mark opposite the minute hand on my Rolex.

And a black mark opposite the hour hand.

The demon was still there when I looked up, still spread-eagled against the wall, still wearing that knowing grin.

I moved around him, waved my hand before his face.

When I touched him he felt like marble.

Time had stopped, but the demon had remained.

I felt sick with relief.

The second hand on my watch was still moving.

I had expected nothing less.

Time had stopped for me—for twenty-four hours of interior time.

If it had been exterior time I'd have been safe—but of course that was too easy.

I'd thought my way into this mess.

I should be able to think my way out, shouldn't I?

I erased the pentagram from the wall, scrubbing until every trace was gone.

Then I drew a new one, using a flexible metal tape to get the lines as straight as possible, making it as large as I could get it in the confined space.

It was still only two feet across.

I left the basement.

I knew where the nearby churches were, though I hadn't been to one in too long.

My car wouldn't start.

Neither would my roommate's motorcycle.

The spell which enclosed me wasn't big enough.

I walked to a Mormon temple three blocks away.

The night was cool and balmy and lovely.

City lights blanked out the stars, but there was a fine werewolf's moon hanging way above the empty lot where the Mormon temple should have been.

I walked another eight blocks to find the B'nai B'rith synagogue and the All Saints church.

All I got out of it was exercise.

I found empty lots.

For me, places of worship didn't exist.

I prayed.

I didn'