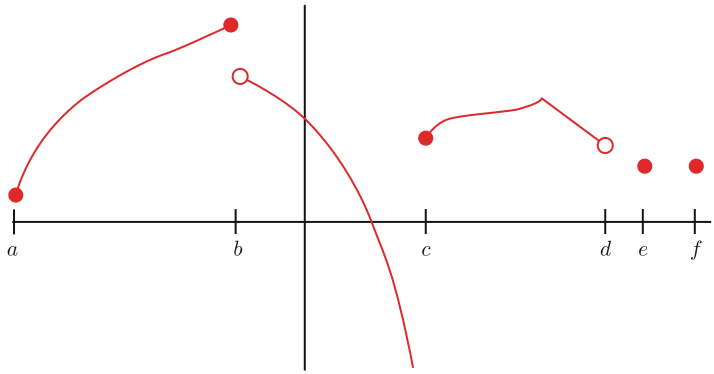

Fig. 4.1 Continuity of a function

only requiring the inequality to hold for those x values satisfying $0 < |x - a| < \delta$ which excludes $x = a$. This restriction is not necessary in the definition of continuity of a function at a point.

Suppose that the point a is in the domain of the function f. Then ***f is continuous at a*** *means that for every $\epsilon > 0$ there exists a $\delta > 0$ such that for every x in the domain of f satisfying $|x - a| < \delta$, it follows that $|f(x) - f(a)| < \epsilon$.*

Notice that the requirement that the point a be an accumulation point of the domain of f has been dropped. As a result, if the function f is defined at an isolated point a, then f is continuous at that point. A function that is not continuous at the point a is **discontinuous at the point** $\boldsymbol{a}$.

A function f is **continuous on a set** $\boldsymbol{A}$ if it is continuous at each point $a \in A$. The function whose graph appears in Fig. 4.1 is discontinuous at $x = b$ because its limit at $x = b$ does not exist. Similarly, it is discontinuous at $x = c$. It is discontinuous at $x = d$ because it is not defined at that point even though the function has a limit there. The function is continuous on the intervals $[a, b)$, (b, c), and (c, d), and at the points $x = e$ and $x = f$. The function is not continuous on the intervals $[a, b]$ or $[c, d]$.

It is a direct consequence of the definition of continuity that if f is continuous at a point a, and if a is an accumulation point of the domain of f, then the limit of $f(x)$ at a exists and is, in fact, $f(a)$. To prove this you would just need to show that if f satisfies the definition of continuity at a, then f also satisfies the definition of $\lim_{x \to a} f(x) = f(a)$. Writing down the definition of continuity gives you that for every $\epsilon > 0$ there is a $\delta > 0$ such that $|x - a| < \delta$ implies $|f(x) - f(a)| < \epsilon$. But if this is true, then certainly $0 < |x - a| < \delta$ implies $|f(x) - f(a)| < \epsilon$, so the definition of limit is satisfied.

PROOF: If the function f is continuous at a, and a is an accumulation point of the domain of f, then $\lim_{x \to a} f(x) = f(a)$.

- Let f be a function continuous at a where a is an accumulation point of the domain of f.
- Given $\epsilon > 0$,
- the definition of continuity says that there is a $\delta > 0$ such that if x is in the domain of f with $|x - a| < \delta$, then $|f(x) - f(a)| < \epsilon$.
- But then if $0 < |x - a| < \delta$, it follows that $|f(x) - f(a)| < \epsilon$ satisfying the definition of $\lim_{x \to a} f(x) = f(a)$.
- Therefore, $\lim_{x \to a} f(x) = f(a)$.

Similarly, if f is defined at a and $\lim_{x \to a} f(x) = f(a)$, then f is continuous at a. Again, the proof of this follows directly from the definitions.

PROOF: If the function f is defined at a and $\lim_{x \to a} f(x) = f(a)$, then f is continuous at a.

- Let f be a function defined at a where $\lim_{x \to a} f(x) = f(a)$.
- Given $\epsilon > 0$,
- the definition of limit says that there is a $\delta > 0$ such that if x is in the domain of f with $0 < |x - a| < \delta$, then $|f(x) - f(a)| < \epsilon$.
- Certainly, if $x = a$, then $|f(x) - f(a)| = |f(a) - f(a)| = 0 < \epsilon$.
- Thus, it follows that $|x - a| < \delta$ implies $|f(x) - f(a)| < \epsilon$ satisfying the definition of f being continuous at a.
- Therefore, f is continuous at a.

4.2 Proving the Continuity of a Function

The template for proofs of $\lim_{x \to a} f(x) = L$ followed directly from the definition of limit. Similarly, a template for proofs of the continuity of a function f at a point a will follow directly from the definition of continuity. Indeed, the definition of continuity requires that for every $\epsilon > 0$ there exist a $\delta > 0$ which satisfies a particular condition. This suggests that a proof of continuity should select an arbitrary $\epsilon > 0$ and proceed to display a value of $\delta > 0$ that causes the needed condition to be satisfied. This is similar to the procedure taken for a limit proof except that the needed condition is slightly different. Thus, here is a template for proofs about the continuity of a function at a point.

TEMPLATE for proving the function f is continuous at the point a

- SET THE CONTEXT: Make statements about what is known about the function f and the numbers a and $f(a)$.
- SELECT AN ARBITRARY ϵ: *Given* $\epsilon > 0$,
- PROPOSE A VALUE FOR δ: *let* δ = __. Here you would insert an appropriate value for δ.
- SELECT AN ARBITRARY x: *Select* x *in the domain of* f *such that* $|x - a| < \delta$.
- LIST IMPLICATIONS: Derive the result $|f(x) - f(a)| < \epsilon$.
- STATE THE CONCLUSION: *Therefore, f is continuous at the point a.*

As a start, consider how to prove that the function defined for all real numbers x as $f(x) = 5x-3$ is continuous at $x = 4$. The proof would begin with "Let $f(x) = 5x-3$. Given $\epsilon > 0, \ldots$." The task is then to find a $\delta > 0$ so that $|f(x)-f(4)| < \epsilon$ for every x satisfying $|x - 4| < \delta$. Working backwards, to get $|f(x) - f(4)| < \epsilon$ one needs $\epsilon > |(5x - 3) - (5 \cdot 4 - 3)| = 5|x - 4|$. Therefore, it seems clear that $|x - 4|$ needs to be less than $\frac{\epsilon}{5}$, so letting $\delta = \frac{\epsilon}{5}$ will work. Note that because $\epsilon > 0$, δ is also greater than 0 as required by the definition of continuity. Putting this into the template results in the following proof.

PROOF: The function $f(x) = 5x - 3$ is continuous at $x = 4$.

- Let $f(x) = 5x - 3$.
- Given $\epsilon > 0$,
- let $\delta = \frac{\epsilon}{5}$ which is greater than 0 since $\epsilon > 0$.
- Select x such that $|x - 4| < \delta = \frac{\epsilon}{5}$.
- Then $\delta > |x-4|$ implies $|f(x)-f(4)| = |(5x-3)-(5\cdot4-3)| = |5x-20| = 5|x - 4| < 5\delta = \epsilon$.
- Therefore, the function f is continuous at 4.

For a more challenging example, consider proving that the function $f(x) = 2x^3 - 4x + 1$ is continuous for all real numbers. This proof not only tackles a more complicated function than the one in the previous example, it is supposed to demonstrate the continuity of the function at the general real number a rather than at a specific value such as $a = 4$. This requires the proof to select an arbitrary a and prove the continuity of f at the point a. By showing that the function is continuous at any arbitrarily chosen a, it shows that the function is continuous at every point a. Again, the proof will select an arbitrary $\epsilon > 0$ and needs to produce a $\delta > 0$ such that $|f(x) - f(a)| < \epsilon$ for all x satisfying $|x - a| < \delta$. The proof needs to select an arbitrary a and an arbitrary $\epsilon > 0$. Does it matter which it does first? In this case where the choice of a does not depend on which ϵ is chosen, and the choice of ϵ does not depend on which a is chosen, the order is not critical. It makes sense to select the a first because you are then challenged to prove that f is continuous at a for which you should choose an $\epsilon > 0$. But since both quantifiers are universal quantifiers (*for all* $a \in \mathbb{R}$ and *for all* $\epsilon > 0$), the order does not matter. If it had been a universal quantifier and an existential quantifier such as "for all $\epsilon > 0$ there exists a $\delta > 0$," then the order would matter a great deal.

Working backwards from $\epsilon > |f(x) - f(a)|$ you can see that you need $\epsilon > |(2x^3 - 4x + 1) - (2a^3 - 4a + 1)| = |2(x^3 - a^3) - 4(x - a)| = |2(x-a)(x^2 + xa + a^2) - 4(x-a)| = |x-a| \cdot |2(x^2 + xa + a^2) - 4|$. You should not be surprised and, in fact, be quite pleased to see that this last expression contains a factor of $|x - a|$ because this will facilitate making the expression small when $|x - a|$ is made small. One only needs to control the size of the other factor $|2(x^2 + xa + a^2) - 4|$. Of course, if x is allowed to wonder too far from a, this other factor could get arbitrarily large, so care must be taken to restrict how far x gets from a. This can be done by requiring that δ not be larger than some conveniently selected value such as 1. That means that $|x - a| < \delta \leq 1$ would imply, for example, that $|x| < |a| + 1$. Given this, there are many ways to find an upper bound for the quantity $|2(x^2 + xa + a^2) - 4|$ where the upper bound does not depend on x. For example, $|2(x^2 + xa + a^2) - 4| \leq 2x^2 + 2|x||a| + 2a^2 + 4 \leq 2(|a| + 1)^2 + 2(|a| + 1)|a| + 2a^2 + 4$. One can afford to be sloppy here and get a simpler looking upper bound by saying $2(|a| + 1)^2 + 2(|a| + 1)|a| + 2a^2 + 4 \leq 2(|a| + 1)^2 + 2(|a| + 1)(|a| + 1) + 2(|a| + 1)^2 + 4(|a| + 1)^2 = 10(|a| + 1)^2$. All you need is an upper bound that depends only on a. This leads to the following proof.

PROOF: The function $f(x) = 2x^3 - 4x + 1$ is continuous on the real numbers.

- Let $f(x) = 2x^3 - 4x + 1$, and let $a \in \mathbb{R}$.
- Given $\epsilon > 0$,
- let $\delta = \min\left(1, \frac{\epsilon}{10(|a|+1)^2}\right)$ which is greater than 0 since 1, ϵ, and $10(|a|+1)^2$ are all positive.
- Select x such that $|x - a| < \delta$. Then $\delta \leq 1$ implies $|x| < |a| + 1$.
- Also, $\delta \leq \frac{\epsilon}{10(|a|+1)^2}$ implies that
  $|f(x) - f(a)| = |(2x^3 - 4x + 1) - (2a^3 - 4a + 1)| = |2(x^3 - a^3) - 4(x - a)| =$
  $|2(x - a)(x^2 + xa + a^2) - 4(x - a)| = |x - a| \cdot |2(x^2 + xa + a^2) - 4| \leq$
  $|x - a| \cdot [2(|a| + 1)^2 + 2(|a| + 1)|a| + 2a^2 + 4] \leq$
  $|x - a| \cdot 2(|a| + 1)^2 + 2(|a| + 1)(|a| + 1) + 2(|a| + 1)^2 + 4(|a| + 1)^2 =$
  $|x - a| \cdot 10(|a| + 1)^2 < \frac{\epsilon}{10(|a|+1)^2} \cdot 10(|a| + 1)^2 = \epsilon.$
- Therefore, the function f is continuous at every real number a.

Not all functions can be expressed with nice formulas. Take, for example, the function $f(x) = \left\{ \begin{array}{cl} 2x & \text{if } x \text{ is rational} \\ x + 1 & \text{if } x \text{ is irrational} \end{array} \right\}$ which behaves differently on the rational numbers than it does on the irrational numbers. Such functions that are defined one way on the rational numbers and another way on the irrational numbers make interesting examples because both the rational and the irrational numbers are **dense** in the real numbers; that is, in every nonempty open interval (a, b), you can find both rational and irrational numbers. For the given function, in every nonempty open interval (a, b) there are values of x where $f(x) = 2x$ and other values of x

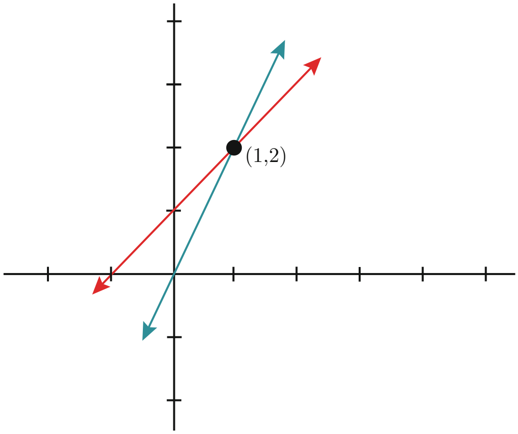

Fig. 4.2 A function equal to $2x$ for rational x (*blue*) and $x + 1$ for irrational x (*red*) The *blue* and *red lines* are not solid

where $f(x) = x + 1$. Indeed, for most real numbers a, $\lim_{x \to a} f(x)$ does not exist. Only at $x = 1$, where $2x$ and $x + 1$ coincide, does this limit exist, and, in fact, at that point $f(x)$ is continuous (Fig. 4.2).

A proof that f is continuous at $x = 1$ would be similar to the two preceding proofs, but you need to be careful to handle $f(x)$ differently depending on whether x is rational or irrational. As in other continuity proofs, given an $\epsilon > 0$ you are faced with producing a value for $\delta > 0$ which will ensure that $|f(x) - f(1)| < \epsilon$ whenever $|x - a| < \delta$. If the function in the proof were equal to $x + 1$ for every value of x, then the value $\delta = \epsilon$ would work because $|x - 1| < \epsilon$ shows that $|f(x) - f(1)| = |(x + 1) - (1 + 1)| = |x - 1| < \epsilon$. If the function in the proof were equal to $2x$ for every value of x, then the value $\delta = \frac{\epsilon}{2}$ would work because $|x - 1| < \frac{\epsilon}{2}$ shows that $|f(x) - f(1)| = |(2x) - (2 \cdot 1)| = 2|x - 1| < \epsilon$. In this proof, then, you can choose $\delta = \min(\epsilon, \frac{\epsilon}{2}) = \frac{\epsilon}{2}$. After selecting an x with $|x - 1| < \delta$, you merely consider two separate cases, one where x is rational, and one where x is irrational. These ideas allow you to produce the following proof.

PROOF: The function $f(x) = \begin{Bmatrix} 2x & \text{if } x \text{ is rational} \\ x + 1 & \text{if } x \text{ is irrational} \end{Bmatrix}$ is continuous at $x = 1$.

- Let $f(x) = \begin{Bmatrix} 2x & \text{if } x \text{ is rational} \\ x + 1 & \text{if } x \text{ is irrational} \end{Bmatrix}$.
- Given $\epsilon > 0$,
- let $\delta = \frac{\epsilon}{2}$ which is greater than 0 since $\epsilon > 0$.
- Select x such that $|x - 1| < \delta = \frac{\epsilon}{2}$.
- If x is a rational number, then $|f(x) - f(1)| = |2x - 2| = 2|x - 1| < 2\delta = \epsilon$.
- If x is an irrational number, then $|f(x) - f(1)| = |(x + 1) - 2| = |x - 1| < \delta < \epsilon$.
- In either case, $|f(x) - f(1)| < \epsilon$.
- Therefore, the function f is continuous at 1.

4.2.1 Exercises

Write proofs of each of the following statements.

1. $f(x) = 4x + 7$ is continuous at $x = -2$.
2. $f(x) = 5x^2 + 3x - 2$ is continuous at $x = 8$.
3. $f(x) = 10x^3 + 25$ is continuous for all real numbers x.
4. $f(x) = |x|$ is continuous at $x = 0$.
5. $f(x) = |x^2 - 9|$ is continuous for all real numbers x.
6. $f(x) = \sqrt{x}$ is continuous for all $x \geq 0$.
7. $f(x) = \sqrt{|x^2 - 4|}$ is continuous for all real numbers x.

4.3 Uniform Continuity

Continuity of a function is a **local property**, that is, whether or not a function f is continuous at a point $x = a$ depends only on how f behaves close to a. In fact, f can be continuous at a and yet have very erratic behavior at points just $\frac{1}{10}$ unit from a or $\frac{1}{100}$ or even $\frac{1}{1{,}000{,}000}$ from a. The last example in the previous section shows a function continuous at $x = 1$ which is continuous for no other value of x. Even if f is continuous at all points of a set A, it could be that proofs of the continuity of f at two points $x = a$ and $x = b$ might need to be quite different. Certainly, there is no reason to believe that, given an $\epsilon > 0$, a value of $\delta > 0$ that works in a proof of the continuity of f at the point a would also work in a proof of the continuity of f at point b.

Consider, for example, the function $f(x) = \frac{1}{x}$ which is continuous for all $x \neq 0$. To prove that f is continuous at $x = 2$, given $\epsilon > 0$ one can use $\delta = \min(1, \epsilon)$ or even be as generous as to let $\delta = \min(1, 2\epsilon)$. But to prove that f is continuous at $x = \frac{1}{2}$ where the function f changes much more rapidly, for the same $\epsilon > 0$, one might need to use $\delta = \min(\frac{1}{4}, \frac{\epsilon}{8})$. You can easily see from the graph of $f(x) = \frac{1}{x}$ that as a gets closer to 0, the $\delta > 0$ chosen for a particular $\epsilon > 0$ will need to get smaller (Fig. 4.3).

Suppose that you wanted to prove that a particular function f was continuous at every a in the domain of f. Such a proof was discussed in the previous section using $f(x) = 2x^3 - 4x + 1$. In that proof, the formula for the $\delta > 0$ chosen for a given $\epsilon > 0$ depended on the point a where f was being shown to be continuous. Clearly, this would have to be the case because f is a cubic function of x which grows increasingly more rapidly as x gets large. But it is not true that every function behaves this way. Some functions change at a constant rate like $f(x) = 6x - 13$ or change at a rate that does not continue to grow such as $f(x) = \frac{1}{x^2+1}$. When writing a proof of the continuity of such functions, it is possible to pick a single value for $\delta > 0$ that depends on $\epsilon > 0$ (as it certainly would have to unless f were constant on each interval in its domain), but where the choice of $\delta > 0$ does not depend on

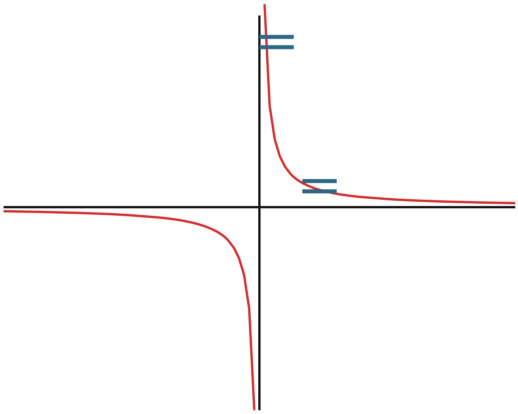

Fig. 4.3 $f(x) = \frac{1}{x}$ is not uniformly continuous

the point a where the continuity needs to be shown. These functions are special and satisfy the following definition. A function f is **uniformly continuous on the set *A*** if for every $\epsilon > 0$ there is a $\delta > 0$ such that $|f(x) - f(y)| < \epsilon$ for every x and y in A satisfying $|x - y| < \delta$. You should compare this definition to the definition of continuity at a point. The difference centers on when the value of $\delta > 0$ needs to be determined. For continuity at a single point, given $\epsilon > 0$, one must specify the value of $\delta > 0$ after being given the value of a but before being given a value for x. Thus, the value of $\delta > 0$ can depend on the value of a even though it cannot depend on the value of x. On the other hand, for uniform continuity, given $\epsilon > 0$, one must specify the value of $\delta > 0$ before learning the values of either x or y, and, therefore, its value cannot depend on either x or y.

The definition of uniform continuity suggests a template for how to prove that a given function f is uniformly continuous on a set A. As in the proof for continuity at a point, you would say that a value for $\epsilon > 0$ has been given. Then you would present a value for $\delta > 0$. Once these two values have been specified, you would need to show that any x and y in A that satisfy $|x-y| < \delta$ also satisfy $|f(x)-f(y)| < \epsilon$. This suggests the following.

TEMPLATE for proving the function *f* is uniformly continuous on the set *A*

- SET THE CONTEXT: Make statements about what is known about the function f.
- SELECT AN ARBITRARY ϵ: *Given* $\epsilon > 0$,
- PROPOSE A VALUE FOR δ: *let* $\delta = \_\_$. Here you would insert an appropriate value for δ.
- SELECT ARBITRARY x and y in A with $|x - y| < \delta$: *Let x and y be in A such that* $|x - y| < \delta$.

(continued)

- LIST IMPLICATIONS: Derive the result $|f(x) - f(y)| < \epsilon$.
- STATE THE CONCLUSION: *Therefore, f is uniformly continuous on the set A.*

Proving the function $f(x) = 6x - 13$ is uniformly continuous on the entire real line is straightforward since the function f changes at a constant rate. This allows you to select a value for $\delta > 0$ based on that rate of change, 6.

PROOF: The function $f(x) = 6x - 13$ is uniformly continuous on the real numbers.

- Let $f(x) = 6x - 13$.
- Given $\epsilon > 0$,
- let $\delta = \frac{\epsilon}{6}$ which is greater than 0 since $\epsilon > 0$.
- Let x and y be real numbers such that $|x - y| < \delta = \frac{\epsilon}{6}$.
- Then $|f(x) - f(y)| = |(6x - 13) - (6y - 13)| = 6|x - y| < 6\delta = \epsilon$.
- Therefore, the function f is uniformly continuous on the real numbers.

Less clear is how to choose a value for $\delta > 0$ when proving $f(x) = \frac{1}{x^2+1}$ is uniformly continuous on the real numbers. To do this, you need to find a way to show $|f(x) - f(y)| < \epsilon$. You would try to find an upper bound for $|f(x) - f(y)| = \left|\frac{1}{x^2+1} - \frac{1}{y^2+1}\right| = \frac{|(y^2+1)-(x^2+1)|}{(x^2+1)(y^2+1)} = \frac{|x+y|}{(x^2+1)(y^2+1)}|x-y|$. This expression is complicated, so it is convenient to find ways to simplify it. The nice thing about working with inequalities rather than equalities is that you are not prevented from making changes that increase the value of your expression. That is, if you can simplify an expression by substituting an expression that is a little larger, that might not be a problem. The numerator in the previous expression is $|x + y|$ which does not simplify algebraically, but it does suggest a possible application of the triangle inequality, $|x + y| \leq |x| + |y|$. Changing $|x + y|$ to $|x| + |y|$ allows the fraction to be broken into two simpler fractions. It allows you to continue with $|f(x) - f(y)| = \frac{|x+y|}{(x^2+1)(y^2+1)}|x - y| \leq \left(\frac{|x|}{(x^2+1)(y^2+1)} + \frac{|y|}{(x^2+1)(y^2+1)}\right)|x - y| \leq \left(\frac{|x|}{x^2+1} + \frac{|y|}{y^2+1}\right)|x - y|$. When $|x| < 1$, you can conclude that $|x| < 1 \leq x^2 + 1$. When $|x| \geq 1$, you can conclude that $|x| \leq x^2 < x^2 + 1$. In either case $\frac{|x|}{(x^2+1)} \leq \frac{x^2+1}{x^2+1} = 1$. This lets you state that $|f(x) - f(y)| = \frac{|x+y|}{(x^2+1)(y^2+1)}|x-y| \leq \left(\frac{|x|}{(x^2+1)(y^2+1)} + \frac{|y|}{(x^2+1)(y^2+1)}\right)|x-y| \leq 2|x - y|$. This suggests that $\delta = \frac{\epsilon}{2}$ will work in the proof.

PROOF: The function $f(x) = \frac{1}{x^2+1}$ is uniformly continuous on the real numbers.

- Let $f(x) = \frac{1}{x^2+1}$.
- Given $\epsilon > 0$,
- let $\delta = \frac{\epsilon}{2}$ which is greater than 0 since $\epsilon > 0$.
- Let x and y be real numbers such that $|x - y| < \delta = \frac{\epsilon}{2}$.
- Then $|f(x) - f(y)| = \left|\frac{1}{x^2+1} - \frac{1}{y^2+1}\right| = \frac{|(y^2+1)-(x^2+1)|}{(x^2+1)(y^2+1)} = \frac{|x+y|}{(x^2+1)(y^2+1)}|x-y| \le \left(\frac{|x|}{(x^2+1)(y^2+1)} + \frac{|y|}{(x^2+1)(y^2+1)}\right)|x-y| \le \left(\frac{|x|}{x^2+1} + \frac{|y|}{y^2+1}\right)|x-y|$
- Note that if $|x| < 1$, then $|x| < x^2 + 1$, and if $|x| \ge 1$, then $|x| \le x^2 < x^2 + 1$.
- In either case, $|x| < x^2 + 1$, so $\frac{|x|}{x^2+1} < 1$, and similarly, $\frac{|y|}{y^2+1} < 1$.
- It follows that $|f(x) - f(y)| \le \left(\frac{|x|}{x^2+1} + \frac{|y|}{y^2+1}\right)|x-y| < 2|x-y| < 2\delta = \epsilon$.
- Therefore, the function f is uniformly continuous on the real numbers.

One of the most memorable theorems from Calculus is the **Mean Value Theorem** which states that if the function f is continuous on the interval $[a, b]$ and differentiable on the interval (a, b), then there is a $c \in (a, b)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$. If the function f has a **bounded derivative** on the interval $[a, b]$, that is, if there is a positive real number M such that $|f'(x)| \le M$ for all values of $x \in [a, b]$, then one can easily see that f is uniformly continuous on that interval. Indeed, if x and y are in $[a, b]$, then there is a c between x and y such that $|f(x)-f(y)| = |f'(c)| \cdot |x-y| \le M \cdot |x-y|$. This implies that given $\epsilon > 0$, the value $\delta = \frac{\epsilon}{M} > 0$ can be used in a proof that f is uniformly continuous on $[a, b]$ for then $|x - y| < \delta$ implies $|f(x) - f(y)| = |f'(c)| \cdot |x - y| < M \cdot |x - y| < M\delta = \epsilon$. This is summarized by saying that a function with a bounded derivative on an interval is uniformly continuous there.

Whenever you learn of the truth of a conditional statement such as the one at the end of the previous paragraph (bounded derivative implies uniform continuity), it is natural to ask whether the converse of the statement is also true (uniform continuity implies bounded derivative). The answer to this particular question is “no, not all functions uniformly continuous on an interval have bounded derivatives there.” In particular, the function $f(x) = |x|$ is an example of a function uniformly continuous on the entire real line, yet it fails to be differentiable at $x = 0$. The function $f(x) = \sqrt{x}$ is uniformly continuous for $x \ge 0$, but its derivative is unbounded near $x = 0$. A more complex example is the function defined by $f(x) = x^2 \sin\left(\frac{1}{x^2}\right)$ when $x \ne 0$ and $f(0) = 0$. This function is uniformly continuous on the interval $[-10, 10]$ even though its derivative, which exists on the entire real line, is not bounded as x approaches 0.

Because the function $f(x) = \sqrt{x}$ has an increasingly large rate of change as x approaches 0, proving that the function is uniformly continuous for $x \ge 0$ provides an interesting challenge. The proof will need to conclude that $\epsilon > |f(x) - f(y)| =$

$|\sqrt{x}-\sqrt{y}| = \frac{|\sqrt{x}-\sqrt{y}|\cdot(\sqrt{x}+\sqrt{y})}{\sqrt{x}+\sqrt{y}} = \frac{|x-y|}{\sqrt{x}+\sqrt{y}}$. As expected, there is a factor of $|x-y|$ in this expression, so that you can try to make the expression small by restricting the size of $|x-y|$. This is easy if the denominator of the expression, $\sqrt{x}+\sqrt{y}$, does not get too small. The problem is if x and y get close to 0, the denominator of the expression will also get close to 0. At first this seems like a significant roadblock. But this roadblock presents its own resolution for if $\sqrt{x}+\sqrt{y}$ is very small, it must certainly be that $|\sqrt{x}-\sqrt{y}|$ is even smaller which is the conclusion that you want. In other words, there are two cases: either $\sqrt{x}+\sqrt{y}$ is small which would imply that $|f(x)-f(y)|$ is small, or $\sqrt{x}+\sqrt{y}$ is large which would imply that $|f(x)-f(y)| = \frac{|x-y|}{\sqrt{x}+\sqrt{y}}$ is small. You only need to decide what to use as the dividing line between "large" and "small." A natural choice would be ϵ itself because $\sqrt{x}+\sqrt{y} < \epsilon$ implies $|\sqrt{x}-\sqrt{y}| < \epsilon$. If $\sqrt{x}+\sqrt{y} \geq \epsilon$, then $|f(x)-f(y)| = \frac{|x-y|}{\sqrt{x}+\sqrt{y}} \leq \frac{|x-y|}{\epsilon}$ which suggests letting $\delta = \epsilon^2$ so that $|x-y| < \delta$ gives $|f(x)-f(y)| < \frac{\epsilon^2}{\epsilon} = \epsilon$. The complete proof follows.

PROOF: The function $f(x) = \sqrt{x}$ is uniformly continuous on the interval $x \geq 0$.

- Let $f(x) = \sqrt{x}$.
- Given $\epsilon > 0$,
- let $\delta = \epsilon^2$ which is greater than 0 since $\epsilon \neq 0$.
- Let x and y be nonnegative real numbers such that $|x-y| < \delta$.
- In the case that $\sqrt{x}+\sqrt{y} < \epsilon$, it follows that $|f(x)-f(y)| = |\sqrt{x}-\sqrt{y}| \leq \sqrt{x}+\sqrt{y} < \epsilon$.
- In the case that $\sqrt{x}+\sqrt{y} \geq \epsilon$, it follows that $|f(x)-f(y)| = |\sqrt{x}-\sqrt{y}| = \frac{|\sqrt{x}-\sqrt{y}|\cdot(\sqrt{x}+\sqrt{y})}{\sqrt{x}+\sqrt{y}} = \frac{|x-y|}{\sqrt{x}+\sqrt{y}} \leq \frac{|x-y|}{\epsilon} < \frac{\delta}{\epsilon} = \frac{\epsilon^2}{\epsilon} = \epsilon$.
- In either case, $|x-y| < \delta$ implies that $|f(x)-f(y)| < \epsilon$, so the function f is uniformly continuous on the interval $x \geq 0$.

There is an important lesson to be learned from this example. When planning how to write a proof, you can pursue one line of thinking which may solve the problem in most but not all cases. Sometimes the special cases where the argument does not work are enough to cause you to abandon your original line of reasoning altogether. But often you can just break your argument into two or more cases and find other techniques to handle the special cases where the original argument does not work.

4.3.1 Exercises

Write proofs of each of the following statements.

1. $f(x) = 3x + 11$ is uniformly continuous on the set of real numbers.
2. $f(x) = -14x + 5$ is uniformly continuous on the set of real numbers.
3. $f(x) = |x|$ is uniformly continuous on the set of real numbers.

4. $f(x) = 8x^2$ is uniformly continuous on the interval $[-6, 6]$.
5. $f(x) = \frac{4}{5x+1}$ is uniformly continuous for $x \geq 0$.
6. $f(x) = \sqrt[3]{x}$ is uniformly continuous on the set of real numbers.
7. $f(x) = x^2$ is not uniformly continuous on the set of real numbers.
8. $f(x) = \frac{1}{x^2}$ is not uniformly continuous on the set $(0, 1)$.

4.4 Compactness and the Heine–Borel Theorem

4.4.1 Open Covers and Subcovers

Let a and b be real numbers with $a < b$. It turns out that if a function f is continuous on the closed interval $[a, b]$, then f is uniformly continuous on that interval. How might you prove this result? As a first try, you might say that for each $\epsilon > 0$ and for each $y \in [a, b]$ there is a $\delta > 0$ such that if $x \in [a, b]$ with $|x - y| < \delta$, then $|f(x) - f(y)| < \epsilon$. Then, having produced a value for δ for each $y \in [a, b]$, you might want to pick the smallest of all of those δ's and hope that this minimum δ would be sufficiently small to work for every $y \in [a, b]$. Unfortunately, you started out with an infinite collection of δ's, each greater than 0. Such an infinite set might not have a minimum value. The set of such δ's is certainly nonempty and bounded below, so the collection does have a greatest lower bound, but that greatest lower bound could be 0, too small to use for the δ in the proof. A finite set of positive numbers always has a minimum value that is positive, but an infinite set of positive numbers might have a greatest lower bound of 0.

Suppose that T is a collection of open intervals, and $A \subseteq \mathbb{R}$. If the set A is contained in the union of the open intervals in T, that is, if $A \subseteq \bigcup_{(s,t)\in T} (s, t)$, then T is called an **open cover of** $\boldsymbol{A}$. A subset $T' \subseteq T$ which is also an open cover of A is called a **subcover of** $\boldsymbol{A}$. In the above suggested proof that the continuity of f on $[a, b]$ implies the uniform continuity of f on $[a, b]$, the definition of continuity at each point of $y \in [a, b]$ produced a collection of open intervals which form an open cover T of $[a, b]$. If that open cover had a *finite* subcover T', then you would be dealing with only a finite number of $\delta > 0$ values, and you could expect to produce a smallest such $\delta > 0$. Whether such a finite subcover exists has nothing to do with the continuous function f that motivated this discussion. A closed bounded interval $[a, b]$ in the real numbers is **compact** which means that every open open cover of $[a, b]$ contains a finite subcover. The fact that every closed bounded interval in the real numbers is compact is known as the Heine–Borel Theorem, and it is central to proving the above result about continuous functions on closed bounded intervals being uniformly continuous there. In fact, the Heine–Borel Theorem is an important tool for proving many results in analysis.

Suppose that for every rational number in $[0, 1]$ you represent the rational number in lowest terms as $\frac{p}{q}$. Then for each of these rational numbers you associate the open interval $(\frac{4p-1}{4q}, \frac{4p+1}{4q})$. For example, the number $\frac{2}{7}$ would be associated with the open interval $(\frac{7}{28}, \frac{9}{28})$. Since the set of rational numbers in $[0, 1]$ is infinite, this collection of open intervals is also infinite. The collection forms an open cover of $[0, 1]$. One possible finite subcover is the collection of intervals associated with rational numbers $\frac{0}{1}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}$, and $\frac{1}{1}$ giving the intervals $(-\frac{1}{4}, \frac{1}{4}), (\frac{3}{16}, \frac{5}{16}), (\frac{3}{12}, \frac{5}{12}), (\frac{3}{8}, \frac{5}{8}), (\frac{7}{12}, \frac{9}{12})(\frac{11}{16}, \frac{13}{16})$, and $(\frac{3}{4}, \frac{5}{4})$. You should verify that these intervals are in the original open cover and do produce the claimed finite subcover. On the other hand, if you associate with each natural number $n > 1$ the open interval $(\frac{1}{n}, 1)$, you get an open cover of the set $(0, 1)$, yet no finite subset of this collection of intervals can cover the entire interval $(0, 1)$. Indeed, any finite collection will only cover the interval $(\frac{1}{m}, 1)$ for some natural number $m > 1$. Since these intervals form an open cover of $(0, 1)$ which does not have a finite subcover, the set $(0, 1)$ is not a compact set.

4.4.2 Proofs of the Heine–Borel Theorem

Presented next are two quite different proofs of the Heine–Borel Theorem. The techniques used in both proofs are instructive, and it is interesting to see how a single result can be proved using two completely different strategies. Given in each case are real numbers $a < b$ and a set of open intervals T that forms an open cover of the closed bounded interval $[a, b]$. Both proofs seek to show that there must be a finite subset of T that covers $[a, b]$. The strategy in the first proof suggests that, whether or not you can cover $[a, b]$ with a finite number of open intervals, you can certainly cover some of the interval starting at a and working at least part of the way toward b. The proof proposes looking at the set

$$S = \{x \in [a, b] \mid T \text{ has a finite subcover that covers the interval } [a, x]\}.$$

The proof first shows that S is not empty because it contains the point a. The set S is bounded above by b, so S has a least upper bound, r. This is not to say that $r \in S$, but if r is not in S, there must be values in S that are arbitrarily close to r. Certainly r is in $[a, b]$, so there is an open interval from T that covers r. Since there are values of S arbitrarily close to r, there are some inside this open interval containing r. This open interval then extends the finite subcover to values greater than r. One can only conclude that r must be b, and, in fact, $b \in S$. Thus, $[a, b]$ has a finite subcover, and the proof is complete (Fig. 4.4).

PROOF (Heine–Borel Theorem): Let $a < b$ be two real numbers, and let T be an open cover of $[a, b]$. Then T contains a finite subcover of $[a, b]$.

- Let $a < b$ be two real numbers, and let T be an open cover of $[a, b]$.
- Define set $S = \{x \in [a, b] \mid T$ has a finite subcover that covers the interval $[a, x]\}$.
- The set T is an open cover of $[a, b]$, and $a \in [a, b]$, so T must contain at least one open interval, (p, q) which contains the point a, that is, $p < a < q$. Since the interval $[a, a]$ is covered by $(p, q) \in T$, the point $a \in S$, and S is not an empty set.
- The set S is bounded above by b.
- Since S is nonempty and bounded above, it has a least upper bound r.
- Since r must be at least a and cannot be greater than b, $r \in [a, b]$, so there is an interval (p, q) in T which contains the point r, that is, $p < r < q$.
- Since $p < r$ and r is the least upper bound of S, p is not an upper bound of S. Thus, there is a point $y \in S$ with $p < y$. This means that there is a finite set of intervals in T that covers $[a, y]$.
- Let $z = \min(\frac{r+q}{2}, b)$. Since $z \geq r$ and $z \in (p, q)$, adding the interval (p, q) to the finite set of intervals of T that covers $[a, y]$ produces a finite set of intervals in T that covers $[a, z]$, and $z \in S$.
- But r is the least upper bound for S, implying that $z \leq r$. Because $z = \min(\frac{r+q}{2}, b)$ and $\frac{r+q}{2} > r$, it must be that $z = b$.
- Because $z \in S$, it follows that $b \in S$ which completes the proof of the theorem.

The second proof of the Heine–Borel Theorem is a proof by contradiction. It begins as the first proof by assuming that $a < b$ are real numbers, and that the interval $[a, b]$ has an open cover T. Then it makes the additional assumption that no finite collection of intervals in T can cover $[a, b]$. This will lead to a contradiction. This proof is not one that the beginning student is likely to invent on their own unless they have seen the technique before.

First, the proof sets $a_0 = a$ and $b_0 = b$ so that the interval $[a_0, b_0] = [a, b]$. Let $m_0 = \frac{a_0+b_0}{2}$ be the midpoint of $[a_0, b_0]$. It must be the case that at least one of the intervals $[a_0, m_0]$ or $[m_0, b_0]$ cannot be covered by a finite number of intervals in T because, if both can be covered by a finite number of intervals, putting those two collections together would give a finite collection of intervals that covered the entire interval $[a_0, b_0] = [a, b]$ contradicting the assumption that this could not be done.

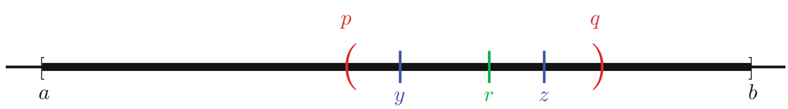

Fig. 4.4 Heine–Borel Theorem first proof

So, if it is the case that $[a_0, m_0]$ cannot be covered by a finite number of intervals in T, let $a_1 = a_0$ and $b_1 = m_0$. Otherwise, if $[m_0, b_0]$ cannot be covered by a finite number of intervals in T, let $a_1 = m_0$ and $b_1 = b_0$. In either case, the new interval $[a_1, b_1] \subseteq [a, b]$ cannot be covered by a finite collection of intervals in T.

Now the proof continues iteratively. If for some $j > 0$, there is an interval $[a_j, b_j]$ contained in $[a, b]$ which cannot be covered by any finite collection of intervals in T, let $m_j = \frac{a_j+b_j}{2}$ be the midpoint of the interval. Either $[a_j, m_j]$ or $[m_j, b_j]$ cannot be covered by a finite collection of intervals from T, so if $[a_j, m_j]$ cannot be covered by a finite collection of intervals, let $a_{j+1} = a_j$ and $b_{j+1} = m_j$. Otherwise, let $a_{j+1} = m_j$ and $b_{j+1} = b_j$. In either case $[a_{j+1}, b_{j+1}]$ cannot be covered by a finite collection of intervals from T. Notice that this process constructs a sequence of intervals $[a_0, b_0], [a_1, b_1], [a_2, b_2], \ldots$ contained in $[a, b]$, none of which can be covered by a finite collection of intervals in T. Also note that $a = a_0 \le a_1 \le a_2 \le \ldots$ while $b = b_0 \ge b_1 \ge b_2 \ge \ldots$, and for each j, the length of the jth interval is $b_j - a_j = \frac{b-a}{2^j}$. Since each a_j term is less than all of the b_k terms, both of the monotone sequences are bounded and, therefore, converge. Moreover, since for each k, $\lim_{j\to\infty} b_j - \lim_{j\to\infty} a_j \le b_k - a_k = \frac{b-a}{2^k}$, it follows that $\lim_{j\to\infty} a_j = \lim_{j\to\infty} b_j = r \in [a, b]$. Note that since the sequence of a_j's increases to r, and the sequence of b_j's decrease to r, the limit $r \in [a_j, b_j]$ for each j. Because the limit, r, is in $[a, b]$, there is an open interval $(p, q) \in T$ such that $r \in (p, q)$. The distance the limit r is from the boundary of the interval (p, q) is $\epsilon = \min(r - p, q - r) > 0$. Since $\lim_{j\to\infty} \frac{b-a}{2^j} = 0$, you can select a j so that $\frac{b-a}{2^j} < \epsilon$. Then it follows that $p \le r - \epsilon < a_j \le r \le b_j \le r + \epsilon < q$, and, so, $[a_j, b_j] \subseteq (p, q)$. But this shows that $[a_j, b_j]$ is covered by the single open interval $(p, q) \in T$ contradicting the fact that $[a_j, b_j]$ could not be covered by a finite collection of intervals in T. Thus, you must conclude that the assumption that $[a, b]$ cannot be covered by a finite number of intervals is false. A formal proof follows (Fig. 4.5).

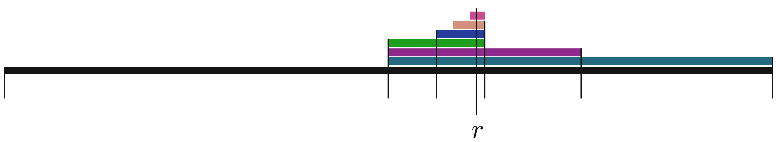

Fig. 4.5 Heine–Borel Theorem second proof

PROOF (Heine–Borel Theorem): Let $a < b$ be two real numbers, and let T be an open cover of $[a, b]$. Then T contains a finite subcover of $[a, b]$.

- Let $a < b$ be two real numbers, and let T be an open cover of $[a, b]$.
- Assume that T contains no finite subcover of $[a, b]$.
- Let $a_0 = a$ and $b_0 = b$ so that the interval $[a_0, b_0] = [a, b]$, and note that no finite collection of intervals in T will cover $[a_0, b_0]$.
- Define sequences $<a_j>$ and $<b_j>$ inductively. For $j \geq 0$, let $[a_j, b_j] \subseteq [a, b]$ be an interval which cannot be covered by a finite collection of open intervals in T, and where $b_j - a_j = \frac{b-a}{2^j}$.
- Let $m_j = \frac{a_j + b_j}{2}$ be the midpoint of $[a_j, b_j]$.
- It must be the case that at least one of the intervals $[a_j, m_j]$ or $[m_j, b_j]$ cannot be covered by a finite number of intervals in T because, if both can be covered by a finite number of intervals, putting those two collections together would give a finite collection of intervals that covered the entire interval $[a_j, b_j]$.
- If $[a_j, m_j]$ cannot be covered by a finite collection of intervals, let $a_{j+1} = a_j$ and $b_{j+1} = m_j$. Otherwise, let $a_{j+1} = m_j$ and $b_{j+1} = b_j$. In either case $[a_{j+1}, b_{j+1}]$ cannot be covered by a finite collection of intervals from T, and $b_{j+1} - a_{j+1} = \frac{\frac{b-a}{2^j}}{2} = \frac{b-a}{2^{j+1}}$.
- Thus, there are monotone sequences $a = a_0 \leq a_1 \leq a_2 \leq \ldots$ and $b = b_0 \geq b_1 \geq b_2 \geq \ldots$, and for each j, the length of the $[a_j, b_j]$ interval is $b_j - a_j = \frac{b-a}{2^j}$.
- Since each a_j term is less than all of the b_k terms, both of the monotone sequences are bounded and, therefore, converge. The fact that $\lim_{j\to\infty} a_j \leq \lim_{j\to\infty} b_j \leq \lim_{j\to\infty} (a_j + \frac{b-a}{2^j})$, shows that $\lim_{j\to\infty} a_j = \lim_{j\to\infty} b_j = r \in [a, b]$.
- Because the limit, r, is in $[a, b]$, there is an open interval $(p, q) \in T$ such that $r \in (p, q)$.
- The distance the limit r is from the boundary of the interval (p, q) is $\epsilon = \min(r - p, q - r) > 0$. Since $\lim_{j\to\infty} \frac{b-a}{2^j} = 0$, there is a j such that $\frac{b-a}{2^j} < \epsilon$.
- It follows that $p \leq r - \epsilon \leq a_j \leq r \leq b_j \leq r + \epsilon < q$, and, so, $[a_j, b_j] \subseteq (p, q)$.
- But then $[a_j, b_j]$ is covered by the single open interval $(p, q) \in T$ contradicting the fact that $[a_j, b_j]$ could not be covered by a finite collection of intervals in T.
- Thus, the assumption that $[a, b]$ cannot be covered by a finite number of intervals is false, and the theorem is proved.

The fact that the interval $[a, b]$ in the Heine–Borel Theorem is both closed and bounded is crucial. The interval $[1, \infty)$ is covered by the collection of open intervals $(j, j + 2)$ for $j = 0, 1, 2, 3, \ldots$, but no finite collection of these open intervals can cover $[1, \infty)$. The interval $(0, 5)$ is covered by the collection $(\frac{1}{j}, 5)$ for $j = 1, 2, 3, 4, \ldots$, but, again, no finite collection of these open intervals can cover $(0, 5)$.