One of the most important properties of monotone sequences is that monotone increasing sequences that are bounded above must converge and monotone decreasing sequences that are bounded below must converge. Thus, bounded monotone sequences converge. If a monotone sequence does not converge, then its terms must continue to grow without bound and approach plus or minus infinity.

So how would you prove that a monotone increasing sequence that is bounded above converges? When proving a limit of the form $\lim_{n\to\infty} a_n = L$, you can work with the inequality $\epsilon > |a_n - L|$ in order to find an appropriate value of $N$ that allows you to use the definition of limit to complete the proof. But in this case, you do not have a general expression for the terms $a_n$, and you have not been given a value for $L$. Somehow you need to use the only known facts about $<a_n>$, that is, the fact that the sequence is both monotone increasing and bounded, to come up with a candidate to serve as the limit, $L$, in the proof.

The definition of a sequence being *bounded above* holds the key. That definition says that the sequence $<a_n>$ is bounded above if the set $\{a_n \mid n \in \mathbb{N}\}$ is bounded above, so there is a real number $M$ which is greater than or equal to each term of the sequence. Will this $M$ be the limit of the sequence? Well, not usually. If $M$ is an upper bound for the sequence, then so are $M + 1$, $M + 100$, and $M + 20{,}000$. They are all upper bounds, but they cannot all be limits of the sequence. You should recognize that the terms of the sequence must get close to the limit, and the only upper bound of the set $\{a_n \mid n \in \mathbb{N}\}$ that the terms could get close to is the *least upper bound* of the set. Since $\{a_n \mid n \in \mathbb{N}\}$ is both nonempty and bounded above, the Completeness Axiom for the real numbers guarantees that such a least upper bound exists. This gives you a candidate for $L$.

The proof will require you to show that for all $n$ greater than some $N$, the terms of the sequence, $<a_n>$, are within $\epsilon$ of $L$. How can this be arranged? Here is where you can use the fact that the sequence is monotone increasing because once you find a single term, $a_n$, that gets within $\epsilon$ of $L$, all the terms that come after this term in the sequence will necessarily have to be between $a_n$ and $L$, so they also will be within $\epsilon$ of $L$. How do you find one term, $a_n$, within $\epsilon$ of $L$? This follows from the fact that $L$ is a least upper bound of $\{a_n \mid n \in \mathbb{N}\}$. Because $L$ is the least upper bound, $L - \epsilon$ being less than the least upper bound, $L$, is not an upper bound, so there must be an element of the set $\{a_n \mid n \in \mathbb{N}\}$ greater than $L - \epsilon$. This gives all the tools needed for the proof (Fig. 3.6).

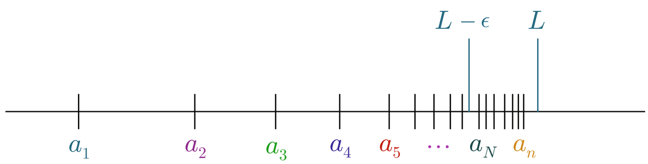



**Fig. 3.6** Proving bounded monotone sequences converge

So how would you write the proof? Certainly the proof would begin with selecting a generic sequence and making a statement about the properties the sequence is assumed to have, that is, its being monotone increasing and bounded above. Then, the proof would proceed to justify the existence of the least upper bound for the set of terms of the sequence; that will give you the target value of $L$. Then, as with most proofs about limits, it would select a value for $\epsilon > 0$. Unlike the limit proofs earlier in this chapter, one cannot immediately state a value for $N$. The existence of $N$ must be proved as discussed in the previous paragraph. Finally, the properties of the sequence can be brought together to show $|a_n - L| < \epsilon$ for all $n > N$. Here is one possible proof.

**PROOF: A monotone increasing sequence that is bound above converges.**

- Let $<a_j>$ be a monotone increasing sequence of real numbers that is bounded above.
- Since the set of terms $A = \{a_j \mid j \in \mathbb{N}\}$ contains $a_1$, it is nonempty, and since it is bounded above, the Completeness Axiom guarantees that $A$ has a least upper bound, $L$.
- Given $\epsilon > 0$, the number $L - \epsilon$ is less than $L$. Since $L$ is the least upper bound of $A$, $L - \epsilon$ is not an upper bound of $A$. Thus, there is an $N \in \mathbb{N}$ such that the term $a_N$ is in $A$ and is larger than $L - \epsilon$.
- Select an $n > N$.
- Because $<a_j>$ is monotone increasing, $a_n \geq a_N$. Because $L$ is an upper bound for $A$, $a_n \leq L$. Therefore, $L - \epsilon < a_N \leq a_n \leq L$, and $|a_n - L| < |(L - \epsilon) - L| = \epsilon$.
- This proves that the sequence $<a_j>$ has limit $L$ and that $<a_j>$ converges.

Note that the proof needs to refer to the sequence $<a_n>$ as well as a particular element of the sequence $a_n$. It could be confusing to the proof reader to use the variable $n$ in both contexts here, especially since the sequence notation $<a_n>$ is used after the choice of a specific value of $n$ is made. That is the reason the proof changed to using the variable $j$ to refer to a generic term index. Then, it could refer to a specific term using index $n$ without confusing the two uses.

There is also a theorem stating that a monotone decreasing sequence that is bounded below converges. The proof of this is left as an exercise.

As an illustration of the usefulness of the above result, consider a sequence defined recursively by $a_1 = 2$, and for $n \geq 1$, $a_{n+1} = \sqrt{a_n + 12}$. That is, $a_1 = 2$, $a_2 = \sqrt{a_1 + 12} = \sqrt{14}$, $a_3 = \sqrt{\sqrt{14} + 12}$, and so forth. One can prove that this sequence converges by showing that the sequence is both monotone increasing and bounded above. Indeed, both of these facts can be established by **mathematical induction**. The reader is likely already familiar with proofs by mathematical induction, but this is an appropriate opportunity to review the method and its merits.

Suppose the variable $n$ represents any natural number, and there is a statement $S(n)$ that includes this variable as part of the statement. For example, the statement could be $\lim_{x\to a} x^n = a^n$. Mathematical induction is a proof technique that uses the following proof template to show that $S(n)$ is true for all $n$ greater than or equal to some base value $b \in \mathbb{N}$.

**TEMPLATE for using mathematical induction to prove the statement $S(n)$ is true for all natural numbers $n \geq b$.**

- SET THE CONTEXT: *The statement will be proved by mathematical induction on $n$ for all $n \geq b$.*
- PROVE $S(b)$: Prove that the statement is true when the variable $n$ is equal to the base value, $b$.
- STATE THE INDUCTION HYPOTHESIS: *Assume that $S(n)$ is true for some natural number $n = k \geq b$.*
- PERFORM THE INDUCTION STEP: Using the fact that $S(k)$ is true, prove that $S(k + 1)$ is true.
- STATE THE CONCLUSION: *Therefore, by mathematical induction, $S(n)$ is true for all natural numbers $n \geq b$.*

It is important to understand that the technique of mathematical induction works. That is, if the statement $S(b)$ is true, and if the statement $S(k) \to S(k + 1)$ is true, then, in fact, $S(n)$ must be true for all natural numbers $n \geq b$. Certainly, $S(b)$ is true. Because $S(b)$ is true, and $S(k) \to S(k + 1)$ is true for all $k \geq b$, it follows that $S(b) \to S(b+1)$, so $S(b+1)$ is true. Then $S(b+1) \to S(b+2)$, $S(b+2) \to S(b+3)$, and so forth, so the fact that $S(n)$ is true for all $n \geq b$ follows.

The strength of mathematical induction is that it is often much easier to provide a proof for the one step $S(k) \to S(k + 1)$ than it is to prove $S(n)$ in the general case. The reader has likely seen many statements proved by mathematical induction while studying Algebra, Calculus, or just about any other branch of mathematics.

Mathematical induction is an excellent tool for proving that the previously introduced recursive sequence is both monotone increasing and bounded above. Clearly, $a_2 = \sqrt{14} > \sqrt{4} = 2 = a_1$ so $a_1 < a_2$. Suppose that for some $k \geq 1$ one has $a_k < a_{k+1}$. Then it follows that $a_k+12 < a_{k+1}+12$ so $\sqrt{a_k + 12} < \sqrt{a_{k+1} + 12}$ which shows that $a_{k+1} < a_{k+2}$. Thus, by mathematical induction it follows that $a_n < a_{n+1}$ for all $n$, and the sequence is monotone increasing. Also clear is that $a_1 = 2 < 4$. Suppose that for some $k \geq 1$ that $a_k < 4$. Then $a_{k+1} = \sqrt{a_k + 12} < \sqrt{4 + 12} = \sqrt{16} = 4$. Thus, by mathematical induction it follows that $a_n < 4$ for all $n$, and the sequence is bounded above. The limit of this sequence can be shown to be 4. In particular, if the limit is $L$, one can conclude that $\sqrt{a_n + 12}$ should be converging to $\sqrt{L + 12}$ which should equal the limit of $a_n$ which is also $L$. Thus, one would expect that $L = \sqrt{L + 12}$. This equation has only one positive real solution, $L = 4$.

### 3.5.7 Cauchy Sequences

A **Cauchy sequence** is a sequence whose terms get close together. As with the definition of *limit*, the concept of "close" needs to be made precise. As with the definition of *limit*, "close" means that given any tolerance $\epsilon > 0$, one can go out far enough in the sequence to ensure that all terms of the sequence beyond that point are within $\epsilon$ of each other. Thus, a sequence is Cauchy if for every $\epsilon > 0$ there is an $N$ such that if natural numbers $m$ and $n$ are both greater than $N$, then $|a_m - a_n| < \epsilon$.

If a sequence of real numbers converges, then the sequence is Cauchy. The proof of this fact uses a strategy employed repeatedly in Analysis, that is, *if two quantities are very close to the same value, then they must be very close to each other*. This standard technique for proving that two quantities are close to each other involves the use of the *triangle inequality*. In particular, if $\lim_{j\to\infty} a_j = L$, then for every $\epsilon > 0$ there is an $N$ such that if natural number $n > N$, then $|a_n - L| < \epsilon$. Well then, certainly if $m$ and $n$ are both natural numbers greater than $N$, then both $|a_m - L| < \epsilon$ and $|a_n - L| < \epsilon$. Adding these two inequalities together shows that $|a_m - L| + |a_n - L| < \epsilon + \epsilon$. The triangle inequality states that for any real numbers $x$ and $y$, $|x| + |y| \geq |x + y|$. Thus, $2\epsilon > |a_m - L| + |a_n - L| = |a_m - L| + |L - a_n| \geq |(a_m - L) + (L - a_n)| = |a_m - a_n|$. Of course, the definition of *Cauchy sequence* requires you to show that $|a_m - a_n|$ is less than $\epsilon$, not $2\epsilon$. But you have an enormous amount of flexibility when working with these types of inequalities, so you could have asked instead for an $N$ such that for all natural numbers $n$ greater than $N$, you have $|a_n - L|$ less than $\frac{\epsilon}{2}$ rather than less than $\epsilon$. Thus, the proof could be as follows.

**PROOF: Every convergent sequence is Cauchy.**

- Let $<a_j>$ be a sequence of real numbers with $\lim_{j\to\infty} a_j = L$.
- Let $\epsilon > 0$ be given.
- From the definition of limit, there is a number $N$ such that for all natural numbers $j > N$, it follows that $|a_j - L| < \frac{\epsilon}{2}$.
- Then for all natural numbers $m$ and $n$ greater than $N$, $|a_m - L| < \frac{\epsilon}{2}$ and $|a_n - L| < \frac{\epsilon}{2}$, so $\epsilon = \frac{\epsilon}{2} + \frac{\epsilon}{2} > |a_m - L| + |a_n - L| = |a_m - L| + |L - a_n| \geq |(a_m - L) + (L - a_n)| = |a_m - a_n|$.
- This shows that the convergent sequence $<a_j>$ is Cauchy.

Note that the converse of this theorem also holds. That is, any sequence of real numbers that is Cauchy is a convergent sequence. This result will be proved in Sect. 3.7. An important and useful consequence of the above theorem is its contrapositive: *If a sequence is not Cauchy, then it does not converge.* Often when one wants to show that a sequence does not converge, one shows that there is some $\epsilon > 0$ such that for every $N$ there are natural numbers $m$ and $n$ greater than $N$ for which $|a_m - a_n| \geq \epsilon$.

Another important property of Cauchy sequences is that all Cauchy sequences are bounded. If the sequence $<a_n>$ is Cauchy, then there is a natural number $N$ such that whenever $m, n \geq N$, the difference $|a_m - a_n| < 1$. The set $\{a_1, a_2, a_3, \ldots, a_N\}$

is a finite set, so it is bounded by some number, $K$. That is, $|a_n| \leq K$ for all $n \leq N$. If $m > N$, then, since both $N$ and $m$ are greater than or equal to $N$, it follows that $|a_m - a_N| < 1$ from which it follows that $|a_m| < |a_N| + 1 \leq K + 1$. Then the sequence $<a_n>$ is necessarily bounded above by $K + 1$ and below by $-(K + 1)$, and the sequence is bounded. A complete proof follows.

**PROOF: All Cauchy sequences are bounded.**

- Let $<a_n>$ be a Cauchy sequence.
- Then there is a natural number $N$ such that for all $m, n \geq N$, $|a_m - a_n| < 1$.
- The set $\{a_1, a_2, a_3, \ldots, a_N\}$ is a finite set, so there is a $K$ such that the set is bounded above by $K$ and bounded below by $-K$.
- Let $m$ be any natural number. If $m \leq N$, then $|a_m| \leq K$. If $m > N$, then $|a_m - a_N| < 1$, so $|a_m| = |a_m - a_N + a_N| \leq |a_m - a_N| + |a_N| < 1 + K$.
- It follows that all terms of the sequence lie between $-(K + 1)$ and $K + 1$, and, thus, the sequence is bounded.

One consequence of the last two results is that since all convergent sequences are Cauchy, all convergent sequences are bounded. The concept of a Cauchy sequence is not only applied to sequences of numbers but also to much more general sequences such as sequences of vectors, sequences of functions, and sequences of linear operators. Of course, one would need a way to discuss distances between the terms of a sequence in these other contexts, but when that makes sense, the concept of a Cauchy sequence becomes important.

### 3.5.8 Exercises

1. Which of the following sequences are monotone? Which of them are bounded above? Which of them are bounded below? Which of them are bounded?

   (a) $a_n = (-1)^n$
   (b) $a_n = \frac{n}{n+1}$
   (c) $a_n = 5^n$
   (d) $a_n = 5^{n^{(-1)^n}}$
   (e) $a_n = \frac{1+(-1)^n}{n+n^{-1}}$
   (f) $a_n = 5 - n(-1)^n$
   (g) $a_n = 1 \cdot \frac{1}{2} \cdot \frac{1}{3} \cdots \frac{1}{n}$

2. Write proofs of each of the following limits.

   (a) $\lim\limits_{n \to \infty} \frac{6n}{3n+1} = 2$
   (b) $\lim\limits_{n \to \infty} \frac{4n-1}{n+6} = 4$
   (c) $\lim\limits_{n \to \infty} \frac{n^2+2n+1}{n^2-2n-5} = 1$

3. If $a_1 = 3$ and $a_n$ is defined recursively by $a_{n+1} = \sqrt{3a_n + 10}$, show that the sequence $<a_n>$ converges.
4. If $a_1 = 7$ and $a_n$ is defined recursively by $a_{n+1} = \sqrt{3a_n + 4}$, show that the sequence $<a_n>$ converges.
5. Prove that a monotone decreasing sequence that is bounded below converges.
6. Let $<a_n>$ be any sequence. Prove that $<a_n>$ has a monotone subsequence.
7. Prove that if $<a_n>$ is a sequence such that $L = \lim_{n\to\infty} a_{2n} = \lim_{n\to\infty} a_{2n+1}$, then the sequence converges to $L$.
8. Prove that if $<a_n>$ is a sequence that converges to $L$, then the sequence $a_1, a_1, a_2, a_2, a_3, a_3, \ldots$ also converges to $L$.
9. Prove that if $<a_n>$ is a sequence that converges to $L$, then the sequence $a_1, a_2, a_2, a_3, a_3, a_3, a_4, a_4, a_4, a_4, \ldots$ also converges to $L$.

## 3.6 Proving That a Limit Does Not Exist

### 3.6.1 *Why a Limit Might Not Exist*

$\lim_{x\to a} f(x) = L$ means that if $x$ is required to stay close to $a$, then $f(x)$ will stay close to $L$. So what does it mean for $\lim_{x\to a} f(x)$ not to exist? Intuitively, it could mean that in every neighborhood of $a$ there are values of $x$ for which $f(x)$ is close to one value $L_1$ and other values of $x$ for which $f(x)$ is close to another value $L_2$. That is what happens with the function $f(x) = \left\{ \begin{array}{ll} 4x - 5 & \text{if } x < 2 \\ 10 - 2x & \text{if } x \geq 2 \end{array} \right\}$ as $x$ approaches 2. For some values of $x$ near 2, $f(x)$ is close to 3, and for some values of $x$ near 2, $f(x)$ is close to 6. Thus, the limit does not exist. Another well-known example is $f(x) = \sin\left(\frac{1}{x}\right)$ which oscillates wildly as $x$ approaches zero, and in every neighborhood of 0, the function takes on all values in the interval $[-1, 1]$ infinitely often. Another way for the limit not to exist is for the values of $f(x)$ to grow without bound and approach infinity or negative infinity such as what happens to $f(x) = \frac{x+3}{(x-5)^2}$ as $x$ approaches 5.

One can write a proof showing that a particular function has no limit at $x = a$, but before discussing how to do this, it is worth taking a close look at the definition of limit.

### 3.6.2 *Quantifiers and Negations*

To say that a function $f$ has a limit at $x = a$ is to say that there exists a real number $L$ such that for all $\epsilon > 0$ there is a $\delta > 0$ such that for every $x$, $0 < |x - a| < \delta$ implies $|f(x) - L| < \epsilon$. This definition is actually a fairly complicated statement. At the heart of it is the conditional statement "$0 < |x - a| < \delta$ implies $|f(x) - L| < \epsilon$."

But this is an open statement, that is, even though the function $f$ and the limit point $a$ are supposedly known, the statement contains variables $x$, $L$, $\epsilon$, and $\delta$, all of which are unknown. Thus, this open statement does not have any truth value until these four variables have been stipulated. They are stipulated with four phrases: "there is a real number $L$," "for all $\epsilon > 0$," "there is a $\delta > 0$," and "for every $x$." These four phrases are called **quantifications** of the variables because they indicate for which values of the variables the following statement must hold. Two of the phrases use the **existential quantifier** "there exists." It indicates that there is at least one value of the variable that will make the following statement true. The other two phrases use the **universal quantifier** "for all." It indicates that every possible value of that variable will make the following statement true. So

- The statement "there exists a real number $L$ such that for all $\epsilon > 0$ there is a $\delta > 0$ such that for every $x$, $0 < |x - a| < \delta$ implies $|f(x) - L| < \epsilon$" begins with the existential quantifier "there exists a real number $L$," and the entire statement is true if, in fact, there is a value of the variable $L$ that makes the following statement true, that is, "for all $\epsilon > 0$ there is a $\delta > 0$ such that for every $x$, $0 < |x - a| < \delta$ implies $|f(x) - L| < \epsilon$."
- The statement "for all $\epsilon > 0$ there is a $\delta > 0$ such that for every $x$, $0 < |x-a| < \delta$ implies $|f(x) - L| < \epsilon$" begins with the universal quantifier "for all $\epsilon > 0$," and the entire statement is true if, in fact, every possible positive value of the variable $\epsilon$ makes the following statement true, that is, "there is a $\delta > 0$ such that for every $x$, $0 < |x - a| < \delta$ implies $|f(x) - L| < \epsilon$."
- The statement "there is a $\delta > 0$ such that for every $x$, $0 < |x - a| < \delta$ implies $|f(x) - L| < \epsilon$" begins with the existential quantifier "there is a $\delta > 0$," and the entire statement is true if, in fact, there is a positive value of the variable $\delta$ that makes the following statement true, that is, "for every $x$, $0 < |x - a| < \delta$ implies $|f(x) - L| < \epsilon$."
- The statement "for every $x$, $0 < |x - a| < \delta$ implies $|f(x) - L| < \epsilon$" begins with the universal quantifier "for every $x$," and the entire statement is true if, in fact, every possible value of the variable $x$ makes the following statement true, that is, "$0 < |x - a| < \delta$ implies $|f(x) - L| < \epsilon$."

A proof that no limit exists must prove the negation of the statement that says that a limit does exist, so it is important that one can generate the negation of a statement that contains quantifiers such as this one does. The logic of doing this is not hard to follow. Suppose the $P(y)$ is a statement that depends on the value of a variable $y$. Then the universally quantified statement "for every $y$, $P(y)$" says that $P(y)$ is true for every possible value of $y$. The negation of "for every $y$, $P(y)$" must be that it is false that every value of $y$ makes $P(y)$ true, so there must be at least one $y$ that makes $P(y)$ a false statement. This means that the negation of "for every $y$, $P(y)$" is the statement "there is a $y$ such that $\neg P(y)$." To negate a universally quantified statement, change the universal quantifier to an existential quantifier and negate the statement that follows.

What if the original statement is an existentially quantified statement such as "there is a $y$ such that $P(y)$?" This statement says that some value of $y$ makes

$P(y)$ true. The negation of this statement must be that no value of $y$ makes $P(y)$ true which is to say that every value of $y$ makes $P(y)$ a false statement. This means that the negation of "there is a $y$ such that $P(y)$" is the statement "for all $y$, $\neg P(y)$." To negate an existentially quantified statement, change the existential quantifier to a universal quantifier and negate the statement that follows.

The statement that $f$ has a limit at $x = a$ is a statement that has an existential quantifier followed by a universal quantifier followed by an existential quantifier followed by a universal quantifier followed by a conditional statement. To prove that $f$ does not have a limit at $x = a$ requires a proof of the negation of that statement. From the previous discussion it is now clear that to get the negation of the statement that $f$ has a limit at $a$, you must flip the two existential quantifiers to universal quantifiers, flip the two universal quantifiers to existential quantifiers, and end with the negation of the conditional statement. The result is "for all real numbers $L$ there is an $\epsilon > 0$ such that for all $\delta > 0$ there is an $x$ such that $0 < |x - a| < \delta$ and $|f(x) - L| \geq \epsilon$."

### *3.6.3 Proving No Limit Exists*

Getting back to writing a proof that a limit does not exist, the proof would need to show that for every real number $L$ there is an $\epsilon > 0$ such that for every $\delta > 0$ there is an $x$ within $\delta$ of $a$ such that $|f(x) - L| \geq \epsilon$. This is often done by exhibiting an $x_1$ and an $x_2$ within $\delta$ of $a$ such that $f(x_1)$ and $f(x_2)$ are so far apart that they could not both be within $\epsilon$ of any $L$. That suggests the following template for proving that a particular limit does not exist.

**TEMPLATE for proving $\lim_{x \to a} f(x)$ does not exist**

- SET THE CONTEXT: Make statements about what is known about the function $f$ and the number $a$.
- SELECT AN ARBITRARY LIMIT $L$: *Given* $L \in \mathbb{R}$,
- PROPOSE A VALUE FOR $\epsilon$: *let* $\epsilon =$ __. Here you would insert a value for $\epsilon$.
- SELECT AN ARBITRARY $\delta > 0$: *Select* $\delta > 0$.
- SELECT VALUES FOR $x_1$ AND $x_2$: *Let* $x_1 =$ __ *and* $x_2 =$ __. *Note that* $0 < |x_1 - a| < \delta$, $0 < |x_2 - a| < \delta$, *and* $|f(x_1) - f(x_2)| \geq 2\epsilon$. You would have selected appropriate $x_1$ and $x_2$ in such a way that $|f(x_1) - f(x_2)|$ exceeds $2\epsilon$.
- LIST IMPLICATIONS: *Assume that* $|f(x_1) - L| < \epsilon$ *and* $|f(x_2) - L| < \epsilon$. *Then* $2\epsilon = \epsilon + \epsilon > |f(x_1) - L| + |f(x_2) - L| = |f(x_1) - L| + |L - f(x_2)| \geq |f(x_1) - L + L - f(x_2)| = |f(x_1) - f(x_2)|$.
- STATE THE CONTRADICTION: *This shows that* $2\epsilon > |f(x_1) - f(x_2)|$ *which is a contradiction.*
- STATE THE CONCLUSION: *Thus, it cannot hold that both* $|f(x_1) - L| < \epsilon$ *and* $|f(x_2) - L| < \epsilon$, *and the limit does not exist.*

For example, consider the limit of $f(x) = \begin{Bmatrix} 4x - 5 & \text{if } x < 2 \\ 10 - 2x & \text{if } x \geq 2 \end{Bmatrix}$ as $x$ approaches 2. Here the limit from the left is 3, and the limit from the right is 6. Thus, no matter how close $x$ is supposed to be to 2, there will be values $x_1$ and $x_2$ within that required tolerance where $f(x_1)$ is close to 3 and $f(x_2)$ is close to 6. If $f(x_1)$ and $f(x_2)$ are both supposed to be within $\epsilon$ of some limit $L$, then it will follow that $f(x_1)$ and $f(x_2)$ will have to be within $2\epsilon$ of each other. Again, you employ the technique of showing that two quantities close to the same value must be close to each other. In particular, if $x_1$ is chosen to be less than 2, $f(x_1)$ will be less than 3. If $x_2$ is chosen to be between 2 and $2\frac{1}{2}$, $f(x_2)$ will be greater than 5. In this case it would be impossible to have $f(x_1)$ and $f(x_2)$ within 2 of each other, and, therefore, it would be impossible to have them both within $\epsilon = 1$ of some limit $L$. This suggests that you will get a contradiction if you set $\epsilon = 1$. Indeed, if a $\delta > 0$ is chosen, you can let $x_1 = 2 - \frac{\delta}{2}$ (that is, less than 2 but within $\delta$ of 2), and let $x_2 = \min\left(2 + \frac{\delta}{2}, 2 + \frac{1}{2}\right)$ (that is, greater than 2 but within $\delta$ of 2 and not so large that $f(x)$ is less than 5). The point of all of this is that now, no matter what value is chosen for $L$, $f(x_1)$ and $f(x_2)$ are more than 2 apart, so how could they both be within 1 of $L$? Specifically, if $|f(x_1) - L| < 1$ and $|f(x_2) - L| < 1$, it follows from the triangle inequality that $2 = 1 + 1 > |f(x_1) - L| + |f(x_2) - L| = |f(x_1) - L| + |L - f(x_2)| \geq |f(x_1) - L + L - f(x_2)| = |f(x_1) - f(x_2)|$ showing $2 > |f(x_1) - f(x_2)|$ which cannot hold. Here is the complete proof (Fig. 3.7).

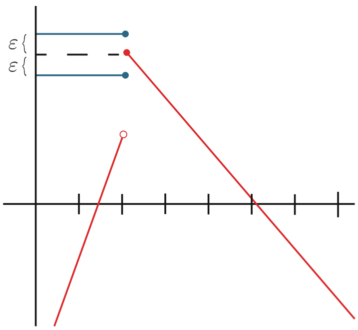


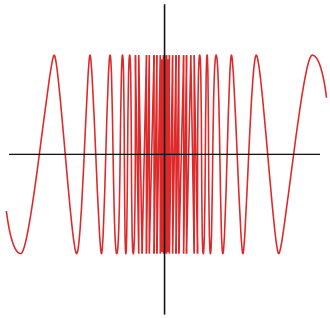

**Fig. 3.7** $f$ has no limit at $x = 2$

**PROOF: The function $\begin{Bmatrix} 4x - 5 & \text{if } x < 2 \\ 10 - 2x & \text{if } x \geq 2 \end{Bmatrix}$ has no limit as $x \to 2$.**

- Let $f(x) = \begin{Bmatrix} 4x - 5 & \text{if } x < 2 \\ 10 - 2x & \text{if } x \geq 2 \end{Bmatrix}$.
- Given any value for $L$, let $\epsilon = 1$, and let $\delta > 0$ be given.
- Let $x_1 = 2 - \frac{\delta}{2}$ and $x_2 = \min\left(2 + \frac{\delta}{2}, 2 + \frac{1}{4}\right)$.
- Note that $0 < |x_1 - 2| < \delta$ and $0 < |x_2 - 2| < \delta$.
- Since $x_1 < 2$, it follows that $f(x_1) < 3$. Since $x_2 > 2$ and $x_2 < 2\frac{1}{4}$, it follows that $f(x_2) > 5$. As a consequence $|f(x_1) - f(x_2)| = f(x_2) - f(x_1) > 5 - 3 = 2$.
- If $|f(x_1) - L| < \epsilon = 1$ and $|f(x_2) - L| < \epsilon = 1$, it would follow that $2 = 1 + 1 > |f(x_1) - L| + |f(x_2) - L| = |f(x_1) - L| + |L - f(x_2)| \geq |f(x_1) - L + L - f(x_2)| = |f(x_1) - f(x_2)| > 2$. This shows that $2 > 2$ which is a contradiction.
- Thus, it cannot hold that both $|f(x_1) - L| < \epsilon$ and $|f(x_2) - L| < \epsilon$, and the limit does not exist.

It is even easier to show that the function $f(x) = \sin\frac{1}{x}$ has no limit as $x$ approaches 0. This is because for every $\delta > 0$ it is easy to find $x_1$ and $x_2$ between 0 and $\delta$ such that $f(x_1) = 1$ and $f(x_2) = -1$. This makes it impossible to find an $L$ where $|f(x_1) - L| < 1$ and $|f(x_2) - L| < 1$. Thus, the proof follows the given template for proving that a limit does not exist (Fig. 3.8).

**Fig. 3.8** Graph of $\sin\frac{1}{x}$



**PROOF: The function $\sin \frac{1}{x}$ has no limit as $x \to 0$.**

- Let $f(x) = \sin \frac{1}{x}$.
- Given any value for $L$, let $\epsilon = 1$, and let $\delta > 0$ be given.
- Select integer $k > \frac{1}{2\pi\delta}$. Let $x_1 = \frac{2}{(4k+1)\pi}$ and $x_2 = \frac{2}{(4k+3)\pi}$.
- Note that both $x_1$ and $x_2$ are positive and less than $\frac{2}{4k\pi} = \frac{1}{2k\pi} < \delta$,
- $f(x_1) = \sin\left[(2k + \frac{1}{2})\pi\right] = 1$, and $f(x_2) = \sin\left[(2k + \frac{3}{2})\pi\right] = -1$.
- If $|f(x_1) - L| < \epsilon = 1$ and $|f(x_2) - L| < \epsilon = 1$, it would follow that $2 = 1 + 1 > |f(x_1) - L| + |f(x_2) - L| = |f(x_1) - L| + |L - f(x_2)| \geq |f(x_1) - L + L - f(x_2)| = |f(x_1) - f(x_2)| = 2$. This shows that $2 > 2$ which is a contradiction.
- Thus, it cannot hold that both $|f(x_1) - L| < \epsilon$ and $|f(x_2) - L| < \epsilon$, and the limit does not exist.

If the function $f(x)$ is unbounded as $x$ approaches $a$, then there is an even easier template to use for the proof that $f(x)$ has no limit. The idea is that since $f(x)$ is unbounded, for any proposed limit $L$ one can find an $x$ close to $a$ such that $|f(x)| > |L| + 1$. Then the difference $|f(x) - L|$ will be forced to be greater than 1. Consider, for example, the function $f(x) = \frac{x+3}{(x-5)^2}$ as $x$ approaches 5. Given $L$, you will want an $x$ with $\frac{x+3}{(x-5)^2} > L + 1$. But with $x$ within 1 of 5, you could claim that $\frac{x+3}{(x-5)^2} > \frac{1}{(x-5)^2} > \frac{1}{|x-5|}$, so by making $|x - 5| < \frac{1}{|L|+1}$ you will have the inequality that you need. Note that the absolute value function was introduced in $|L| + 1$ to take care of the embarrassing circumstance that $L$ is negative, and in particular, when $L = -1$. The proof is as follows.

**PROOF: The function $\frac{x+3}{(x-5)^2}$ has no limit as $x \to 5$.**

- Let $f(x) = \frac{x+3}{(x-5)^2}$.
- Given any value for $L$, let $\epsilon = 1$, and let $\delta > 0$ be given.
- Select a value of $x$ between 5 and $5 + \min\left(1, \delta, \frac{1}{|L|+1}\right)$.
- Note that $0 < |x - 5| < \delta$
- and $f(x) = \frac{x+3}{(x-5)^2} > \frac{1}{(x-5)^2} > \frac{1}{x-5} > |L| + 1$.
- It follows that $|f(x) - L| > |L| + 1 - L \geq 1$.
- Thus, it cannot hold that $|f(x) - L| < \epsilon$, and the limit does not exist.

### *3.6.4 Exercises*

Write the negation of each of the following statements.

1. There exists $x$ such that $x^2 = A$.
2. For all $x$ there is a $y$ such that $g(x) = f(y)$.

3. There is an integer $k$ such that $f(x) \leq f(k)$ for all $x$ between $k$ and $k+1$.
4. For all $x > 0$ and all $y > 0$ there exists a $z < 0$ such that $f(z) \geq xf(y)$.

Prove that the following limits do not exist.

5. $f(x) = \frac{x}{|x|}$ as $x \to 0$
6. $f(x) = x \sin\left(\frac{1}{x-1}\right)$ as $x \to 1$
7. $f(x) = \left\{\begin{array}{l} 5x \text{ if } x < 3 \\ 4x \text{ if } x \geq 3 \end{array}\right\}$ as $x \to 3$
8. $f(x) = \frac{4}{x^2-4}$ as $x \to 2$

## 3.7 Accumulation Points

A set $A$ has an **accumulation point** $p$ if for every $\epsilon > 0$ there is an $x \in A$ with $x \neq p$ and $|x - p| < \epsilon$. Informally, $p$ is an accumulation point of $A$ if there are points of $A$ that are arbitrarily close to $p$. Note that the fact that $p$ is an accumulation point of the set $A$ has nothing to do with whether $p$ is actually an element of $A$. For example, the set $A = \left\{\frac{1}{n} \mid n \in \mathbb{N}\right\}$ has one accumulation point, 0, because for every $\epsilon > 0$ there is an $n \in \mathbb{N}$ with $\frac{1}{n} < \epsilon$. Here the accumulation point 0 is not an element of the set $A$. The set $B = [0, 4]$ (the closed interval from 0 to 4) has infinitely many accumulation points. Indeed, every element of the interval $B$ is an accumulation point of $B$ because for each $x \in [0, 4]$ and each $\epsilon > 0$ there are infinitely many points in $B$ within $\epsilon$ of $x$. Here all of the accumulation points of $B$ are in $B$. Each point $x \in [0, 4]$ is also an accumulation point of the set $C = (0, 4) \cap \mathbb{Q}$, the set of rational numbers between 0 and 4. Here, some of the accumulation points are in $C$, and some are not. The set of natural numbers, $\mathbb{N}$, has no accumulation points. An element $a$ of a set that is not an accumulation point of that set is called an **isolated point** of the set. For any isolated point $a$, there is an $\epsilon > 0$ such that $a$ is the only element of the set in the interval $(a - \epsilon, a + \epsilon)$ (Fig. 3.9).

A word of warning is needed here. The term *accumulation point* is not used the same way by all authors. Many texts, especially those in Topology, will use the terms *limit point* or *cluster point* instead of *accumulation point*. Even more confusing is that some texts use the term *accumulation point* for something different.


**Fig. 3.9** Set with accumulation point $a$ and isolated point $b$

The first observation to make about accumulation points is that if $p$ is an accumulation point of set $A$, then for every $\epsilon > 0$ there is not only one point of $A$ within $\epsilon$ of $p$ but infinitely many points of $A$ within $\epsilon$ of $p$. The definition of accumulation point guarantees at least one point of $A$ within $\epsilon$ of $p$, but once one point, $x \in A$, is found to be within $\epsilon$ of $p$, the definition also says that there must be another point $y \in A$ with $0 < |y - p| < |x - p|$. Since for each $x \in A$ close to $p$ there must be another point $y \in A$ even closer to $p$, it follows that there are infinitely many points of $A$ within $\epsilon$ of $p$.

Perhaps the most used fact about accumulation points is known as the Bolzano–Weierstrass Theorem which states that every infinite bounded set of real numbers has an accumulation point. As pointed out earlier, $\mathbb{N}$ has no accumulation points, and it is an infinite set. But $\mathbb{N}$ is not a bounded set. Intuitively, one cannot have a bounded infinite set without an accumulation point because one runs out of places to put the infinite number of points. If the points of a set are not allowed to bunch up anywhere, then one will not be able to find room for infinitely many of the points within a bounded interval.

There are several good strategies used to prove the Bolzano–Weierstrass Theorem, and two of those strategies are presented here. Of course, one only needs one good strategy to prove a theorem, but these proofs are instructive and use techniques commonly employed in Analysis proofs. One begins each proof with a statement about the set $A$ being an infinite bounded set. Since $A$ is a bounded set, it will have a lower bound, $a$, and an upper bound, $b$ showing that $A \subseteq [a, b]$. The first strategy is to construct the set $S = \{x \geq a \mid [a, x] \cap A \text{ is finite}\}$, that is, a value $x \geq a$ is in the set $S$ if there are finitely many element of $A$ which fall in the interval $[a, x]$. First observe that the set $S$ is an interval. This follows because if $y \in S$, then $[a, y] \cap A$ is finite, so if $x$ is between $a$ and $y$, then $[a, x] \cap A \subseteq [a, y] \cap A$ must also be finite, and $x \in A$. The next observation is that $S$ is not empty because the point $a$, whether or not it is in $A$, is in $S$ since $[a, a] \cap A$ contains at most one point, so it is finite. Since $[a, b] \cap A = A$ is an infinite set, the set $S$ is bounded above by $b$. The Completeness Axiom now shows that $S$ must have a least upper bound, $p$. It will follow that $p$ is an accumulation point of $A$ because for all $\epsilon > 0$, the set $A$ will have only finitely many elements less than $p - \epsilon$ but infinitely many elements less than $p + \epsilon$ implying that there are infinitely many elements of $A$ within $\epsilon$ of $p$. Here is the complete proof.