1.7. Bounded sets. The upper bound and the lower bound of a set

We say that a set of real numbers Z is *bounded*, if two numbers m and M exist such that every number x belonging to the set Z satisfies the double inequality $m \leqslant x \leqslant M$. Assuming only that there exists a number M

satisfying the inequality $M \geqslant x$ for every x belonging to the set Z, we call the set Z *bounded above*. Similarly, a set Z is called *bounded below* if a number m exists satisfying the above condition: $m \leqslant x$.

The geometrical meaning of these ideas is the following. If a set is bounded it means that it is contained in a certain segment on the numerical line. A set is bounded above or below, when it is contained in an infinite radius directed to the left or to the right, respectively.

Applying the continuity principle we shall prove the following theorem:

If a non-empty set Z is bounded above, then among the numbers M, satisfying the inequality $M \geqslant x$ for any x belonging to Z, there exists a least one. This number will be called the upper bound of the set Z.

Similarly, *if a set Z is bounded below, then among the numbers m, satisfying the inequality $m \leqslant x$ for any x belonging to Z, there exists a greatest one, which is called the lower bound of the set Z.*

Proof. Let the set Z be bounded above. Let us divide the set of all real numbers in two classes as follows. To the second class there will belong all numbers M satisfying the inequality $M \geqslant x$ for any x belonging to Z. To the first class there will belong all other real numbers; that means that a number a belongs to the first class when in the set Z there exists a number greater than a. Such a division of the set of real numbers in two classes is called a *cut*, i. e. any number belonging to the first class is less than any number belonging to the second one. Indeed, supposing a number M of the second class to be less than a number a of the first class and knowing that a number x exists in the set Z such that $a < x$ we should have $M < x$; but this contradicts the definition of the second class.

Moreover, let us note that both classes are non-empty. Indeed, if a number z belongs to the set Z (and such a number exists, the set Z being non-empty by assumption), then the number $z-1$ belongs to the first class. The second class is also non-empty, since the set Z is bounded above.

According to the continuity principle there exists either a greatest number in the first class or a least number in the second class. However, the first eventuality is not possible. Namely, if a belongs to the first class and $a < x$ (where x belongs to Z), then denoting by a' any number between a and x, e. g. $a' = \frac{a+x}{2}$, we have also $a' < x$, but this means that a' belongs to the first class, too. Thus, to any number a of the first class we may find a number a' greater than a in this class. This means that in the first class a greatest number does not exist. Hence, there exists in the second class a least number, i. e. the least number among the numbers M satisfying the inequality $M \geqslant x$ for every x belonging to the set Z. Hence the theorem is proved.

The proof of existence of a lower bound is completely analogous.

Let us note that the upper bound and the lower bound of a set Z do not necessarily belong to this set. E. g. the bounds of the open interval [1] $a < x < b$ are the numbers a and b which do not belong to this interval.

1.8*. The axiomatic treatment of real numbers

The notion of a real number which we have assumed to be known from the middle-school course may be introduced in an axiomatic way as follows.

We assume that in the set of real numbers two operations can be performed: *addition* $x+y$ and *multiplication* xy. These operations satisfy the laws of *com-*

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[1] By an open interval we understand the set of numbers x

mutativity and *associativity*:

$$x+y=y+x\,,\qquad xy=yx\,,$$
$$(x+y)+z=x+(y+z)\,,\qquad (xy)z=x(yz)\,.$$

Moreover, the multiplication is *distributive* with respect to the addition:

$$x(y+z)=xy+xz\,.$$

Two (different) numbers 0 and 1 are the *moduli* of addition and multiplication respectively, i.e.

$$x+0=x\,,\qquad x\cdot 1=x\,.$$

Further, we assume that in the set of real numbers subtraction and division are always possible, except division by 0. In other words, we assume that to any pair of numbers x and y a number z (called the *difference* $x-y$) exists such that

$$x=y+z$$

and, in the case where $y\neq 0$, a number w (called the *quotient* $x:y$) exists such that

$$x=yw\,.$$

Besides the above axioms concerning the operations, we take following axioms concerning the order relation $x<y$. We assume that any two distinct real numbers x and y are in this relation in one or another direction, i.e. either $x<y$ or $y<x$. This relation is *transitive*, i.e.

the conditions $x<y$ *and* $y<z$ *imply* $x<z$

and *asymmetric*, i.e.

if $x<y$, *then the relation* $y<x$ *does not hold.*

The order relation is connected with the basic operations by the following axiom:

if $y<z$, *then* $x+y<x+z$ *and if, moreover,* $0<x$, *then* $xy<xz$.

(i)–(iii)). Namely, by $R+R'$ we denote the set of all numbers which are sums of two numbers, the first of which belongs to R and the second one belongs to R'. It is easily seen that the set $R+R'$ satisfies the conditions (i)–(iii). Moreover, the axioms concerning the addition (commutativity, associativity etc.) are satisfied. The real number 0 is defined as the set of all negative rational numbers. Similarly, the real number 1 is the set of all rational numbers less than the rational-one. In general, if r is a rational number, then we understand by "the real number r" the set of all rational numbers less than the rational number r (in practice, we identify the rational number r and the real number r). $-R$ means the set of all rational numbers of the form $-x$, where x takes all values which do not belong to R; however, if the set of numbers not belonging to R contains a least number r, then we do not include the number $-r$ in the set $-R$ (that is the case, when R is "the real number r"). One may prove that $R+(-R)=0$.

The multiplication of real numbers will be defined in the following way: if $R\geqslant 0$ and $R'\geqslant 0$, then RR' is the set containing all negative rational numbers and numbers of the form rr', where r is a non-negative number belonging to R and, similarly, r' is a non-negative number belonging to R'. Moreover, we assume

$$(-R)(-R')=RR', \quad (-R)R'=-(RR')=R(-R').$$

It may easily be proved that all the axioms concerning the multiplication are then satisfied.

Finally, the continuity axiom is satisfied. Indeed, let A, B denote a cut in the domain of real numbers. We denote by R the set of rational numbers with the property that r belongs to R if and only if it belongs to one of the real numbers belonging to the class A.

It is proved that the above defined set R satisfies the conditions (i)–(iii), so that it is a real number. Then we prove that R "lies on the cut A, B", i.e. that it is

Finally, the last axiom in the theory of real numbers which we are accepting is the *Dedekind continuity principle* formulated in § 1.5.

All arithmetic and algebraic theorems known from courses in the middle-school may be deduced from the above axioms.

1.9*. Real numbers as sets of rational numbers [1]

The notion of a real number may be defined on the basis of the theory of rational numbers as follows.

Real numbers may be considered to be identical with sets of rational numbers R, satisfying the following conditions:

(i) the set R does not contain a greatest number, i.e. for any number belonging to the set R there exists a greater number in R;

(ii) if a number x belongs to R, then any rational number less than x belongs to R;

(iii) the set R is non-empty and is not equal to the set of all rational numbers.

For real numbers defined in this way we define first of all the relation of order. Namely, we write $R < R'$, when the set R is a part of the set R' (different from R'); or, when the set R' contains numbers which do not belong to R (it is easily seen that for any two sets satisfying the conditions (i)–(iii) always one is contained in the other). Both these definitions are equivalent.

We easily find that the order relation defined in this way satisfies the axioms given in § 1.8, i.e. it is a transitive, asymmetric relation which holds for any pair of different sets R and R' in one or other direction.

Now we define the addition of real numbers, i.e. the addition of sets R and R' (satisfying the conditions

(1) We give here an outline of the so-called Dedekind theory of real numbers.

2.4. Operations on sequences

THEOREM. *Assuming the sequences* $a_1, a_2, \ldots$ *and* $b_1, b_2, \ldots$ *to be convergent, the following four formulae hold* ([1]):

$$\lim(a_n + b_n) = \lim a_n + \lim b_n\,, \tag{6}$$

$$\lim(a_n - b_n) = \lim a_n - \lim b_n\,, \tag{7}$$

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(1) To simplify the symbolism we shall often omit the equality

(8) $\lim(a_n \cdot b_n) = \lim a_n \cdot \lim b_n$,

(9) $\lim \dfrac{a_n}{b_n} = \dfrac{\lim a_n}{\lim b_n}$ (*when* $\lim b_n \neq 0$).

This means that under our assumptions the limit of the sum exists and is equal to the sum of limits, the limit of the difference exists and is equal to the difference of limits etc.

Proof. Let us write $\lim a_n = g$ and $\lim b_n = h$. A number $\varepsilon > 0$ let be given. Hence a number k exists such that the inequalities $|a_n - g| < \varepsilon/2$ and $|b_n - h| < \varepsilon/2$ hold for $n > k$. We add both these inequalities under the sign of absolute value. We obtain

$$|(a_n + b_n) - (g + h)| < \varepsilon .$$

This means that the sequence with general term $c_n = a_n + b_n$ is convergent to the limit $g + h$. Thus, we have proved the formula (6).

In particular, if b_n takes a constant value: $b_n = c$, formulae (6) and (3) imply:

(10) $$\lim(a_n + c) = c + \lim a_n .$$

Now, we shall prove the formula (8). We have to "estimate" the difference $|a_n b_n - gh|$. To be able to apply the convergence of the sequences $a_1, a_2, \ldots$ and $b_1, b_2, \ldots$ we transform this difference as follows:

$$a_n b_n - gh = a_n b_n - a_n h + a_n h - gh = a_n(b_n - h) + h(a_n - g) .$$

Since the sequence $a_1, a_2, \ldots$ is convergent, it is bounded and so a number M exists such that $|a_n| < M$. Applying to the last equation the formulae for the absolute value of a sum and of a product we get:

$$|a_n b_n - gh| \leqslant |a_n(b_n - h)| + |h(a_n - g)|$$
$$\leqslant M \cdot |b_n - h| + |h| \cdot |a_n - g| .$$

Now, let us take a number $\eta > 0$ independently of ε. Hence a number k exists such that we have $|a_n - g| < \eta$

Writing $\eta = \frac{1}{2}\varepsilon h^2$, we get

$$\left|\frac{1}{b_n} - \frac{1}{h}\right| < \varepsilon,$$

whence the formula (13) follows.

The formula (9) follows from (8) and (13):

$$\lim \frac{a_n}{b_n} = \lim a_n \cdot \frac{1}{b_n} = \lim a_n \cdot \lim \frac{1}{b_n} = \frac{\lim a_n}{\lim b_n}.$$

Remarks. (α) We have assumed that the sequences $\{a_n\}$ and $\{b_n\}$ are convergent. This assumption is essential, for it may happen that the sequence $\{a_n + b_n\}$ is convergent, although both the sequences $\{a_n\}$ and $\{b_n\}$ are divergent; then the formula (6) cannot be applied. As an example one can take: $a_n = n$, $b_n = -n$.

However, if the sequence $\{a_n + b_n\}$ and one of the two sequences, e. g. the sequence $\{a_n\}$ are convergent, then the second one is also convergent. For $b_n = (a_n + b_n) - a_n$, and so the sequence $\{b_n\}$ is convergent as a difference of two convergent sequences.

Analogous remarks may be applied to the formulae (7)–(9).

(β) In the definition of a sequence we have assumed that the enumeration of the elements begins with 1. It is convenient to generalize this definition assuming that the enumeration begins with an arbitrary positive integer (and even with an arbitrary integer), e.g. with 2, 3 or another positive integer. So is e.g. in the proof of the formula (13). We have proved that $b_n \neq 0$ beginning with a certain k. Thus, the sequence $\frac{1}{b_n}$ is defined just beginning with this k (for if $b_n = 0$, then $\frac{1}{b_n}$ does not mean any number).

This remark is connected with the following property of sequences, easy to prove: *the change of a finite number of terms of a sequence has influence neither on the con-*

and $|b_n - h| < \eta$ for $n > k$. Thus,

$$|a_n b_n - gh| < M\eta + |h|\eta = (M + |h|)\eta .$$

Till now we have not assumed anything about the positive number η. Let us now assume that $\eta = \varepsilon/(M + |h|)$. So we conclude that the inequality $|a_n b_n - gh| < \varepsilon$ holds for $n > k$. Thus, we have proved the formula (8).

In particular, if we write $b_n = c$ we get

$$\lim(c \cdot a_n) = c \lim a_n , \tag{11}$$

$$\lim(-a_n) = -\lim a_n , \tag{12}$$

where the formula (12) follows from (11) by the substitution $c = -1$.

Formulae (6) and (12) imply the formula (7), for

$$\lim(a_n - b_n) = \lim[a_n + (-b_n)]$$
$$= \lim a_n + \lim(-b_n) = \lim a_n - \lim b_n .$$

Before proceeding to the proof of the formula (9), we shall prove the following special case of this formula:

$$\lim \frac{1}{b_n} = \frac{1}{\lim b_n} \quad (\textit{when } \lim b_n \neq 0) . \tag{13}$$

First, we note that for sufficiently large n the inequality $b_n \neq 0$ holds. We shall prove an even stronger statement: we have $|b_n| > \frac{1}{2}|h|$ for sufficiently large n. Indeed, since $\frac{1}{2}|h| > 0$, a number k exists such that $|b_n - h| < \frac{1}{2}|h|$ for $n > k$. Hence,

$$|h| - |b_n| \leqslant |h - b_n| < \tfrac{1}{2}|h| \quad \text{and thus} \quad |b_n| > \tfrac{1}{2}|h| .$$

To prove the formula (13), the difference

$$\left|\frac{1}{b_n} - \frac{1}{h}\right| = \left|\frac{h - b_n}{h \cdot b_n}\right| = \frac{|h - b_n|}{|h| \cdot |b_n|}$$

has to be estimated.

But for sufficiently large n we have $|h - b_n| < \eta$ and $|b_n| > \frac{1}{2}|h|$, i. e. $1/|b_n| < 2/|h|$. Thus,

$$\left|\frac{1}{b_n} - \frac{1}{h}\right| < \frac{2\eta}{h^2} .$$