$\mathbf{R} \sim E$ is closed. Now suppose $\mathbf{R} \sim E$ is closed. Let x belong to E. Then there must be an open interval that contains x that is contained in E, for otherwise every open interval that contains x contains points in $X \sim E$ and therefore x is a point of closure of $\mathbf{R} \sim E$. Since $\mathbf{R} \sim E$ is closed, x also belongs to $\mathbf{R} \sim E$. This is a contradiction. □

Since $\mathbf{R} \sim [\mathbf{R} \sim E] = E$, it follows from the preceding proposition that *a set is closed if and only if its complement is open.* Therefore, by De Morgan's Identities, Proposition 8 may be reformulated in terms of closed sets as follows.

Proposition 12 *The empty-set $\emptyset$ and $\mathbf{R}$ are closed; the union of any finite collection of closed sets is closed; and the intersection of any collection of closed sets is closed.*

A collection of sets $\{E_\lambda\}_{\lambda \in \Lambda}$ is said to be a **cover** of a set E provided $E \subseteq \bigcup_{\lambda \in \Lambda} E_\lambda$. By a subcover of a cover of E we mean a subcollection of the cover that itself also is a cover of E. If each set E_λ in a cover is open, we call $\{E_\lambda\}_{\lambda \in \Lambda}$ an **open cover** of F. If the cover $\{E_\lambda\}_{\lambda \in \Lambda}$ contains only a finite number of sets, we call it a **finite cover**. This terminology is inconsistent: In "open cover" the adjective "open" refers to the sets in the cover; in "finite cover" the adjective "finite" refers to the collection and does not imply that the sets in the collection are finite sets. Thus the term "open cover" is an abuse of language and should properly be "cover by open sets." Unfortunately, the former terminology is well established in mathematics.

The Heine–Borel Theorem *Let F be a closed and bounded set of real numbers. Then every open cover of F has a finite subcover.*

Proof Let us first consider the case that F is the closed, bounded interval $[a, b]$. Let $\mathcal{F}$ be an open cover of $[a, b]$. Define E to be the set of numbers $x \in [a, b]$ with the property that the interval $[a, x]$ can be covered by a finite number of the sets of $\mathcal{F}$. Since $a \in E$, E is nonempty. Since E is bounded above by b, by the completeness of $\mathbf{R}$, E has a supremum; define $c = \sup E$. Since c belongs to $[a, b]$, there is an $\mathcal{O} \in \mathcal{F}$ that contains c. Since $\mathcal{O}$ is open there is an $\epsilon > 0$, such that the interval $(c - \epsilon, c + \epsilon)$ is contained in $\mathcal{O}$. Now $c - \epsilon$ is not an upper bound for E, and so there must be an $x \in E$ with $x > c - \epsilon$. Since $x \in E$, there is a finite collection $\{\mathcal{O}_1, \ldots, \mathcal{O}_k\}$ of sets in $\mathcal{F}$ that covers $[a, x]$. Consequently, the finite collection $\{\mathcal{O}_1, \ldots, \mathcal{O}_k, \mathcal{O}\}$ covers the interval $[a, c + \epsilon)$. Thus $c = b$, for otherwise $c < b$ and c is not an upper bound for E. Thus $[a, b]$ can be covered by a finite number of sets from $\mathcal{F}$, proving our special case.

Now let F be any closed and bounded set and $\mathcal{F}$ an open cover of F. Since F is bounded, it is contained in some closed, bounded interval $[a, b]$. The preceding proposition tells us that the set $\mathcal{O} = \mathbf{R} \sim F$ is open since F is closed. Let $\mathcal{F}^*$ be the collection of open sets obtained by adding $\mathcal{O}$ to $\mathcal{F}$, that is, $\mathcal{F}^* = \mathcal{F} \cup \mathcal{O}$. Since $\mathcal{F}$ covers F, $\mathcal{F}^*$ covers $[a, b]$. By the case just considered, there is a finite subcollection of $\mathcal{F}^*$ that covers $[a, b]$ and hence F. By removing $\mathcal{O}$ from this finite subcover of F, if $\mathcal{O}$ belongs to the finite subcover, we have a finite collection of sets in $\mathcal{F}$ that covers F. □

We say that a countable collection of sets $\{E_n\}_{n=1}^{\infty}$ is **descending** or **nested** provided $E_{n+1} \subseteq E_n$ for every natural number n. It is said to be **ascending** provided $E_n \subseteq E_{n+1}$ for every natural number n.

The Nested Set Theorem *Let $\{F_n\}_{n=1}^{\infty}$ be a descending countable collection of nonempty closed sets of real numbers for which F_1 bounded. Then*

$$\bigcap_{n=1}^{\infty} F_n \neq \emptyset.$$

Proof We argue by contradiction. Suppose the intersection is empty. Then for each real number x, there is a natural number n for which $x \notin F_n$, that is, $x \in \mathcal{O}_n = \mathbf{R} \sim F_n$. Therefore $\bigcup_{n=1}^{\infty} \mathcal{O}_n = \mathbf{R}$. According to Proposition 4, since each F_n is closed, each $\mathcal{O}_n$ is open. Therefore $\{\mathcal{O}_n\}_{n=1}^{\infty}$ is an open cover of $\mathbf{R}$ and hence also of F_1. The Heine-Borel Theorem tells us that there a natural number N for which $F \subseteq \bigcup_{n=1}^{N} \mathcal{O}_n$. Since $\{F_n\}_{n=1}^{\infty}$ is descending, the collection of complements $\{\mathcal{O}_n\}_{n=1}^{\infty}$ is ascending. Therefore $\bigcup_{n=1}^{N} \mathcal{O}_n = \mathcal{O}_N = \mathbf{R} \sim F_N$. Hence $F_1 \subseteq \mathbf{R} \sim F_N$, which contradicts the assumption that F_N is a nonempty subset of F_1. □

Definition *Given a set X, a collection $\mathcal{A}$ of subsets of X is called a σ-algebra (of subsets of X) provided (i) the empty-set, $\emptyset$, belongs to $\mathcal{A}$; (ii) the complement in X of a set in $\mathcal{A}$ also belongs to $\mathcal{A}$; (iii) the union of a countable collection of sets in $\mathcal{A}$ also belongs to $\mathcal{A}$.*

Given a set X, the collection $\{\emptyset, X\}$ is a σ-algebra which has two members and is contained in every σ-algebra of subsets of X. At the other extreme is the collection of sets 2^X which consists of all subsets of X and contains every σ-algebra of subsets of X. For any σ-algebra $\mathcal{A}$, we infer from De Morgan's Identities that $\mathcal{A}$ is closed with respect to the formation of intersections of countable collections of sets that belong to $\mathcal{A}$; moreover, since the empty-set belongs to $\mathcal{A}$, $\mathcal{A}$ is closed with respect to the formation of finite unions and finite intersections of sets that belong to $\mathcal{A}$. We also observe that a σ-algebra is closed with respect to relative complements since if A_1 and A_2 belong to $\mathcal{A}$, so does $A_1 \sim A_2 = A_1 \cap [X \sim A_2]$. The proof of the following proposition follows directly from the definition of σ-algebra.

Proposition 13 *Let $\mathcal{F}$ be a collection of subsets of a set X. Then the intersection $\mathcal{A}$ of all σ-algebras of subsets of X that contain $\mathcal{F}$ is a σ-algebra that contains $\mathcal{F}$. Moreover, it is the smallest σ-algebra of subsets of X that contains $\mathcal{F}$ in the sense that any σ-algebra that contains $\mathcal{F}$ also contains $\mathcal{A}$.*

Let $\{A_n\}_{n=1}^{\infty}$ be a countable collection of sets that belong to a σ-algebra $\mathcal{A}$. Since $\mathcal{A}$ is closed with respect to the formation of countable intersections and unions, the following two sets belong to $\mathcal{A}$:

$$\limsup\{A_n\}_{n=1}^{\infty} = \bigcap_{k=1}^{\infty}\left[\bigcup_{n=k}^{\infty} A_n\right] \text{ and } \liminf\{A_n\}_{n=1}^{\infty} = \bigcup_{k=1}^{\infty}\left[\bigcap_{n=k}^{\infty} A_n\right].$$

The set $\limsup\{A_n\}_{n=1}^{\infty}$ is the set of points that belong to A_n for countably infinitely many indices n while the set $\liminf\{A_n\}_{n=1}^{\infty}$ is the set of points that belong to A_n except for at most finitely many indices n.

Although the union of any collection of open sets is open and the intersection of any finite collection of open sets is open, as we have seen, the intersection of a *countable* collection of open sets need not be open. In our development of Lebesgue measure and

Since $\mathcal{U}$ is the union of open sets it is open. It has been constructed so that (9) holds. □

The Extreme Value Theorem *A continuous real-valued function on a nonempty closed, bounded set of real numbers takes a minimum and maximum value.*

Proof Let f be a continuous real-valued function on the nonempty closed bounded set E of real numbers. We first show that f is bounded on E, that is, there is a real number M such that

$$|f(x)| \leq M \text{ for all } x \in E. \tag{10}$$

Let x belong to E. Let $\delta > 0$ respond to the $\epsilon = 1$ challenge regarding the criterion for continuity of f at x. Define $I_x = (x - \delta, x + \delta)$. Therefore if x' belongs to $E \cap I_x$, then $|f(x') - f(x)| < 1$ and so $|f(x')| \leq |f(x)| + 1$. The collection $\{I_x\}_{x \in E}$ is an open cover of E. The Heine-Borel Theorem tells us that there are a finite number of points $\{x_1, \ldots, x_n\}$ in E such that $\{I_{x_k}\}_{k=1}^n$ also covers E. Define $M = 1 + \max\{|f(x_1)|, \ldots, |f(x_n)|\}$. We claim that (10) holds for this choice of E. Indeed, let x belong to E. There is an index k such that x belongs to I_{x_k} and therefore $|f(x)| \leq 1 + |f(x_k)| \leq M$. To see that f takes a maximum value on E, define $m = \sup f(E)$. If f failed to take the value m on E, then the function $x \mapsto 1/(f(x) - m)$, $x \in E$ is a continuous function on E which is unbounded. This contradicts what we have just proved. Therefore f takes a maximum value of E. Since $-f$ is continuous, $-f$ takes a maximum value, that is, f takes a minimum value on E. □

The Intermediate Value Theorem *Let f be a continuous real-valued function on the closed, bounded interval $[a, b]$ for which $f(a) < c < f(b)$. Then there is a point x_0 in (a, b) at which $f(x_0) = c$.*

Proof We will define by induction a descending countable collection $\{[a_n, b_n]\}_{n=1}^\infty$ of closed intervals whose intersection consists of a single point $x_0 \in (a, b)$ at which $f(x_0) = c$. Define $a_1 = a$ and $b_1 = b$. Consider the midpoint m_1 of $[a_1, b_1]$. If $c < f(m_1)$, define $a_2 = a_1$ and $b_2 = m_1$. If $f(m_1) \geq c$, define $a_2 = m_1$ and $b_2 = b_1$. Therefore $f(a_2) \leq c \leq f(b_2)$ and $b_2 - a_2 = [b_1 - a_1]/2$. We inductively continue this bisection process to obtain a descending collection $\{[a_n, b_n]\}_{n=1}^\infty$ of closed intervals such that

$$f(a_n) \leq c \leq f(b_n) \text{ and } b_n - a_n = [b - a]/2^{n-1} \text{ for all } n. \tag{11}$$

According to the Nested Set Theorem, $\bigcap_{n=1}^\infty [a_n, b_n]$ is nonempty. Let x_0 belong to $\bigcap_{n=1}^\infty [a_n, b_n]$. Observe that

$$|a_n - x_0| \leq b_n - a_n = [b - a]/2^{n-1} \text{ for all } n.$$

Therefore $\{a_n\} \to x_0$. By the continuity of f at x_0, $\{f(a_n)\} \to f(x_0)$. Since $f(a_n) \leq c$ for all n, and the set $(-\infty, c]$ is closed, $f(x_0) \leq c$. By a similar argument, $f(x_0) \geq c$. Hence $f(x_0) = c$. □

Definition *A real-valued function f defined on a set E of real numbers is said to be* ***uniformly continuous*** *provided for each $\epsilon > 0$, there is a $\delta > 0$ such that for all x, x' in E,*

$$\textit{if } |x - x'| < \delta, \textit{ then } |f(x) - f(x')| < \epsilon.$$