Chapter 2

Direct Integration

Recall that the process of finding an indefinite integral is called *integration*. Just like for differentiation, for integration purposes, along with the *integration rules* (see Theorem 1.1), we need a set of validated or established basic integrals, called the *table of basic integrals*.

2.1. Table Integrals and Useful Integration Formula

For our integration purposes, we use the *Table of Basic Integrals* consisting of twenty-one integrals (see Appendix A). We have already validated most of them in the preceding examples. Observe that there exist much more extensive tables of integrals.

Exercise 2.1 (Validating Table Integrals).
Validate integrals 12 and 14 of the *Table of Basic Integrals* (Appendix A).

We are almost ready to integrate now. However, at the moment, we are in the situation of an awkward deficiency demonstrated by the following examples: knowing the table integrals

$$\int e^x\,dx = e^x + C \quad \text{or} \quad \int \sin x\,dx = -\cos x + C,$$

we do not know how to evaluate their slightest variations such as

$$\int e^{-x}\,dx \quad \text{or} \quad \int \sin(2x+3)\,dx.$$

These are particular cases of the routine situation of evaluating the integral of the form

$$\int f(ax+b)\,dx,$$

where a and b are real coefficients with $a \neq 0$, when $\int f(x)\,dx$ is known, which, as we shall see in Sec. 3.1.4, can be treated by the *Method of Substitution*.
To deal with this very common situation, let us, however, not wait until the *Substitution Method* is developed, but prove and start applying immediately the following

Theorem 2.1 (Useful Integration Formula).
Let a and b be real coefficients with $a \neq 0$. If

$$\int f(x)\,dx = F(x) + C,$$

then

$$\int f(ax+b)\,dx = \frac{1}{a}F(ax+b) + C.$$

Proof. Considering that $F'(x) = f(x)$, by the *Chain Rule* (see, e.g., [1, 6]), we have:

$$\frac{d}{dx}\left[\frac{1}{a}F(ax+b)\right] = \frac{1}{a}F'(ax+b)[ax+b]' = \frac{1}{a}f(ax+b)a = f(ax+b).$$

□

Remarks 2.1 (Useful Integration Formula).

- We intentionally do not indicate the intervals of integration not to obscure the simplicity of the formulation and proof.
- As we see in Sec. 3.1.4, the *Useful Integration Formula* (Theorem 2.1) is a special case of the so-called *trivial substitution*.

Examples 2.1 (Applying the Useful Integration Formula). Hereafter, the boxes contain explanatory formulas and/or text.

1. $\int e^{-x}\,dx$ by the *useful integration formula* with $a = -1$ and $b = 0$;

$$= \frac{1}{(-1)}e^{-x} + C = -e^{-x} + C.$$

2. $\int \sin(2x+3)\,dx$

by the *useful integration formula* with $a = 2$ and $b = 3$;

$$= \frac{1}{2}(-\cos(2x+3)) + C = -\frac{1}{2}\cos(2x+3) + C.$$

3. $\int \sqrt[3]{4-10x}\,dx$ switching to the *power form*;

$$= \int (4-10x)^{1/3}\,dx$$

by the *useful integration formula* with $a = -10$ and $b = 4$;

$$= \frac{1}{(-10)} \cdot \frac{1}{4/3}(4-10x)^{4/3} + C = -\frac{3}{40}(4-10x)^{4/3} + C.$$

4. $\int \frac{1}{x/2+\pi}\,dx$

by the *useful integration formula* with $a = 1/2$ and $b = \pi$;

$$= \frac{1}{1/2}\ln|x/2+\pi| + C = 2\ln|x/2+\pi| + C.$$

Table of Basic Integrals (Appendix A), integrals 17 and 21.

For $a > 0$,

5. $\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a^2}\int \frac{1}{(x/a)^2+1}\,dx$

by the *useful integration formula* with $a = 1/a$ and $b = 0$;

$$= \frac{1}{a^2}\frac{1}{1/a}\arctan\frac{x}{a} + C = \frac{1}{a}\arctan\frac{x}{a} + C \quad \text{on } (-\infty,\infty).$$

6. $\int \frac{1}{x\sqrt{x^2-a^2}}\,dx = \frac{1}{a^2}\int \frac{1}{(x/a)\sqrt{(x/a)^2-1}}\,dx + C$

by the *useful integration formula* with $a = 1/a$ and $b = 0$;

$$= \frac{1}{a^2}\frac{1}{1/a}\operatorname{arcsec}\left|\frac{x}{a}\right| + C = \frac{1}{a}\operatorname{arcsec}\left|\frac{x}{a}\right| + C$$

on each of the intervals $(-\infty,-a)$, (a,∞).

Exercise 2.2 (Deriving Table Integrals).
Apply the *Useful Integration Formula* (Theorem 2.1) to derive integral 19 of the *Table of Basic Integrals* (Appendix A).

When evaluating an integral of the form

$$\int (ax+b)^n\,dx,$$

where a and b are real coefficients and $n = 1, 2, 3, \ldots$ is a natural number, the *Useful Integration Formula* (Theorem 2.1) becomes especially handy for large values of the exponent n, when the use of the *binomial formula* leads to cumbersome computations.

Example 2.2 (Applying the Useful Integration Formula).
Evaluate the integral

$$\int (5x-7)^{99}\,dx.$$

Solution: Using the *binomial formula* (see Appendix C) to expand $(5x-7)^{99}$ would result into 100 terms. Applying the *Useful Integration Formula* (Theorem 2.1) "saves the day":

$$\int (5x-7)^{99}\,dx$$

by the *useful integration formula* with $a = 5$ and $b = -7$;

$$= \frac{1}{5}\frac{1}{100}(5x-7)^{100} + C = \frac{1}{500}(5x-7)^{100} + C.$$

2.2. What Is Direct Integration and How Does It Work?

What Is Direct Integration?

Definition 2.1 (Direct Integration).
By *direct integration*, we understand the process of integration, which, using the *integration rules* (Theorem 1.1) alone, reduces the integral of a given function to a combination of table integrals. Such integration makes no use of any special integration techniques, but may employ the *Useful Integration Formula* (Theorem 2.1) when appropriate.

Remark 2.2. When executing direct integration, before applying the *integration rules*, one may need to manipulate the integrand performing multiplication/division, applying relevant identities of algebra/trigonometry, or using such tricks as transforming products into sums, multiplying and dividing by the conjugate radical expression, or completing the square.

How Does Direct Integration Work?

In the following examples, we consider various scenarios of *direct integration* explaining its execution step-by-step.

2.2.1. *By Integration Rules Only*

Example 2.3 (Integration by the Rules Only).

$$\int (7x^3 + 5e^{-x} + 14)\,dx \qquad \text{by the } \textit{integration rules};$$

$$= 7\int x^3\,dx + 5\int e^{-x}\,dx + 14\int 1\,dx$$

$$\text{by the } \textit{useful integration formula};$$

$$= \frac{7}{4}x^4 - 5e^{-x} + 14x + C.$$

2.2.2. *Multiplication/Division Before Integration*

When evaluating an integral one is to be mindful of the absence of the *product* and *quotient rules* for integration.

Thus, the following "solution"

$$\int \frac{(x-1)(x+2)}{\sqrt{x}}\,dx = \frac{\int (x-1)\,dx \int (x+2)\,dx}{\int \sqrt{x}\,dx} = \cdots$$

is *incorrect.*

To solve correctly, we are to execute multiplication and division before integration.

Examples 2.4 (Multiplication/Division Before Integration).

1. $$\int \frac{(x-1)(x+2)}{\sqrt{x}}\,dx \qquad \text{multiplying and switching to the } \textit{power form};$$

$$= \int \frac{x^2 + x - 2}{x^{1/2}}\,dx \qquad \text{dividing termwise};$$

$$= \int (x^{3/2} + x^{1/2} - 2x^{-1/2})\,dx \qquad \text{by the } \textit{integration rules};$$

$$= \int x^{3/2}\,dx + \int x^{1/2}\,dx - 2\int x^{-1/2}\,dx = \frac{2}{5}x^{5/2} + \frac{2}{3}x^{3/2} - 4x^{1/2} + C.$$

2. $\displaystyle\int \frac{x^3 - 2x^2 + 4x - 5}{x-1}\,dx$

dividing $\boxed{\dfrac{x^3 - 2x^2 + 4x - 5}{x-1} = x^2 - x + 3 - \dfrac{2}{x-1}}$ (*verify*);

$$= \int \left(x^2 - x + 3 - \frac{2}{x-1}\right) dx$$ by the *integration rules*;

$$= \int x^2\,dx - \int x\,dx + 3\int 1\,dx - 2\int \frac{1}{x-1}\,dx$$

by the *useful integration formula*;

$$= \frac{x^3}{3} - \frac{x^2}{2} + 3x - 2\ln|x-1| + C.$$

The latter example represents a particular case of integration of rational functions considered in detail in Chapter 7.

2.2.3. *Applying Minor Adjustments*

By *"minor adjustments"* to the integrand, we understand adding and subtracting or multiplying and dividing by the same constant.

Example 2.5 (Applying Minor Adjustments).

$$\int x\sqrt{5-2x}\,dx$$ multiplying and dividing by -2;

$$= -\frac{1}{2}\int (-2x)(5-2x)^{1/2}\,dx$$ adding and subtracting 5;

$$= -\frac{1}{2}\int (5-2x-5)(5-2x)^{1/2}\,dx = -\frac{1}{2}\int \left[(5-2x)^{3/2} - 5(5-2x)^{1/2}\right] dx$$

by the *integration rules*;

$$= -\frac{1}{2}\int (5-2x)^{3/2}\,dx + \frac{5}{2}\int (5-2x)^{1/2}\,dx$$

by the *useful integration formula*;

$$= -\frac{1}{2}\frac{1}{(-2)}\frac{2}{5}(5-2x)^{5/2} + \frac{5}{2}\frac{1}{(-2)}\frac{2}{3}(5-2x)^{3/2} + C$$

$$= \frac{1}{10}(5-2x)^{5/2} - \frac{5}{6}(5-2x)^{3/2} + C.$$

2.2.4. *Using Identities*

Certain identities of algebra and trigonometry useful for integration can be found in Appendix C.

Examples 2.6 (Using Identities).

1. $\int 2^{5x}3^{-x}\,dx$ by the *exponents laws* (see Appendix C);

$$= \int \left(2^5 \cdot 3^{-1}\right)^x\,dx = \int \left(\frac{32}{3}\right)^x\,dx = \frac{(32/3)^x}{\ln(32/3)} + C.$$

2. $\int \frac{\cos x + 1}{\sin^2 x}\,dx$ dividing termwise and rewriting equivalently;

$$= \int \left[\frac{\cos x}{\sin^2 x} + \frac{1}{\sin^2 x}\right]\,dx = \int \left[\frac{1}{\sin x}\frac{\cos x}{\sin x} + \left(\frac{1}{\sin x}\right)^2\right]\,dx$$

since $\boxed{\frac{1}{\sin x} = \csc x, \quad \frac{\cos x}{\sin x} = \cot x}$ (see Appendix C);

$$= \int \left[\csc x \cot x + \csc^2 x\right]\,dx$$ by the *integration rules*;

$$= \int \csc x \cot x\,dx + \int \csc^2 x\,dx = -\csc x - \cot x + C.$$

When evaluating the trigonometric integral

$$\int \cos^2 x\,dx,$$

in the absence of the *product rule* for integration, we are to use *power-reduction identity*

$$\cos^2\theta = \frac{1 + \cos 2\theta}{2}$$

(see Appendix C) as follows:

Example 2.7 (Using Identities).

$$\int \cos^2 x\,dx$$

by the *power-reduction identity* with $\theta = x$, $\boxed{\cos^2 x = \frac{1 + \cos 2x}{2}}$;

$$= \int \frac{1+\cos 2x}{2}\,dx \qquad \text{by the } \textit{integration rules};$$

$$= \frac{1}{2}\left[\int 1\,dx + \int \cos 2x\,dx\right] \qquad \text{by the } \textit{useful integration formula};$$

$$= \frac{1}{2}\left[x + \frac{1}{2}\sin 2x\right] + C = \frac{1}{2}x + \frac{1}{4}\sin 2x + C.$$

More generally, using a relevant *power-reduction identity* allows us to evaluate integrals of the form

$$\int \cos^2(ax+b)\,dx, \quad \int \sin^2(ax+b)\,dx,$$

where a and b are real coefficients with $a \neq 0$.

Similarly, the trigonometric integral

$$\int \sin 3x \cos x\,dx$$

containing a product is found via the *product-to-sum identity*

$$\sin\alpha\cos\beta = \frac{1}{2}\left[\sin(\alpha-\beta) + \sin(\alpha+\beta)\right]$$

(see Appendix C) as follows:

Example 2.8 (Using Identities).

$$\int \sin 3x \cos x\,dx$$

by the relevant *product-to-sum identity* with $\alpha = 3x$ and $\beta = x$,

$$\boxed{\sin 3x\cos x = \frac{1}{2}\left[\sin(3x-x) + \sin(3x+x)\right] = \frac{1}{2}\left[\sin 2x + \sin 4x\right]};$$

$$= \int \frac{1}{2}\left[\sin 2x + \sin 4x\right]\,dx \qquad \text{by the } \textit{integration rules};$$

$$= \frac{1}{2}\left[\int \sin 2x\,dx + \int \sin 4x\,dx\right] \qquad \text{by the } \textit{useful integration formula};$$

$$= \frac{1}{2}\left[\frac{1}{2}(-\cos 2x) + \frac{1}{4}(-\cos 4x)\right] + C = -\frac{1}{4}\cos 2x - \frac{1}{8}\cos 4x + C.$$

More generally, this approach applies to the integrals of the form

$$\int \cos(ax+b)\cos(cx+d)\,dx,$$
$$\int \sin(ax+b)\sin(cx+d)\,dx,$$
$$\int \sin(ax+b)\cos(cx+d)\,dx,$$

where a, b, c, and d are real coefficients with $a, c \neq 0$.

The following trigonometric integral

$$\int \tan^2 5x\,dx$$

can be evaluated via the *Pythagorean identity*

$$\tan^2\theta = \sec^2\theta - 1$$

(see Appendix C) as follows:

Example 2.9 (Using Identities).

$$\begin{aligned}
&\int \tan^2 5x\,dx \\
&\qquad\qquad \text{by the latter identity with } \theta = 5x,\quad \boxed{\tan^2 5x = \sec^2 5x - 1}; \\
&= \int \left[\sec^2 5x - 1\right]\,dx \qquad \text{by the } \textit{integration rules}; \\
&= \int \sec^2 5x\,dx - \int 1\,dx \qquad \text{by the } \textit{useful integration formula}; \\
&= \frac{1}{5}\tan 5x - x + C.
\end{aligned}$$

More generally, this approach applies to the integrals of the form

$$\int \tan^2(ax+b)\,dx, \quad \int \cot^2(ax+b)\,dx,$$

where a and b are real coefficients with $a \neq 0$.

2.2.5. *Transforming Products into Sums*

As is seen from Example 2.8, in the absence of the product rule for integration, one can attempt transforming a product into a sum. Let us use this approach for the following problem.

Example 2.10 (Transforming Products into Sums).
Evaluate the integral

$$\int \frac{1}{(2x-1)(2x+3)}\,dx.$$

Solution: The integrand is a product

$$\frac{1}{2x-1}\cdot\frac{1}{2x+3}.$$

However, the simple fact that the polynomials $2x-1$ and $2x+3$ share the same nonconstant term $2x$ allows us to rewrite it as follows:

$$\frac{1}{2x-1}\cdot\frac{1}{2x+3}=\frac{1}{4}\left[\frac{1}{2x-1}-\frac{1}{2x+3}\right]$$

by replacing the product with the difference

$$\frac{1}{2x-1}-\frac{1}{2x+3}$$

and scaling the latter by coefficient $1/4$, where the denominator 4 is the difference of the constant terms: $3-(-1)$.
Thus,

$$\int \frac{1}{(2x-1)(2x+3)}\,dx \qquad \text{transforming product into sum:}$$

$$\boxed{\frac{1}{(2x-1)(2x+3)}=\frac{1}{4}\left[\frac{1}{2x-1}-\frac{1}{2x+3}\right]};$$

$$=\int \frac{1}{4}\left[\frac{1}{2x-1}-\frac{1}{2x+3}\right]dx \qquad \text{by the } \textit{integration rules};$$

$$=\frac{1}{4}\left[\int \frac{1}{2x-1}\,dx-\int \frac{1}{2x+3}\,dx\right] \qquad \text{by the } \textit{useful integration formula};$$

$$=\frac{1}{4}\left[\frac{1}{2}\ln|2x-1|-\frac{1}{2}\ln|2x+3|\right]+C=\frac{1}{8}\left[\ln|2x-1|-\ln|2x+3|\right]+C.$$

The prior problem, as well as problem 2 from Examples 2.4, is a special case of integration of rational functions studied in Chapter 7.

Example 2.11 (Table of Basic Integrals (Appendix A), integral 18**).**
For $a>0$,

$$\int \frac{1}{x^2-a^2}\,dx=\frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right|+C$$

on each of the intervals $(-\infty, -a)$, $(-a, a)$, (a, ∞).

$$\int \frac{1}{x^2 - a^2}\, dx$$

transforming product into sum:

$$\boxed{\frac{1}{x^2 - a^2} = \frac{1}{(x-a)(x+a)} = \frac{1}{2a}\left[\frac{1}{x-a} - \frac{1}{x+a}\right]};$$

$$= \int \frac{1}{2a}\left[\frac{1}{x-a} - \frac{1}{x+a}\right] dx$$

by the *integration rules*;

$$= \frac{1}{2a}\left[\int \frac{1}{x-a}\, dx - \int \frac{1}{x+a}\, dx\right]$$

by the *useful integration formula* with $a = 1$ and $b = \pm a$;

$$= \frac{1}{2a}\left[\ln|x-a| - \ln|x+a|\right] + C$$

by the *laws of logarithms* (see Appendix C);

$$= \frac{1}{2a} \ln\left|\frac{x-a}{x+a}\right| + C.$$

2.2.6. *Using Conjugate Radical Expressions*

Example 2.12 (Using Conjugate Radical Expressions).

$$\int \frac{1}{\sqrt{2x+7} - \sqrt{2x+3}}\, dx$$

multiplying and dividing by the *conjugate radical expression*;

$$= \int \frac{\sqrt{2x+7} + \sqrt{2x+3}}{(\sqrt{2x+7} - \sqrt{2x+3})(\sqrt{2x+7} + \sqrt{2x+3})}\, dx$$

simplifying;

$$= \int \frac{\sqrt{2x+7} + \sqrt{2x+3}}{2x+7-(2x+3)}\, dx = \int \frac{1}{4}\left[\sqrt{2x+7} + \sqrt{2x+3}\right] dx$$

switching to the *power form*;

$$= \int \frac{1}{4}\left[(2x+7)^{1/2} + (2x+3)^{1/2}\right] dx$$

by the *integration rules*;

$$= \frac{1}{4}\left[\int (2x+7)^{1/2}\, dx + \int (2x+3)^{1/2}\, dx\right]$$

by the *useful integration formula*;

$$= \frac{1}{4}\frac{1}{2}\frac{2}{3}(2x+7)^{3/2} + \frac{1}{4}\frac{1}{2}\frac{2}{3}(2x+3)^{3/2} + C = \frac{(2x+7)^{3/2}}{12} + \frac{(2x+3)^{3/2}}{12} + C.$$