9. Show that

$$\frac{1}{\sqrt{\{C(Ac-aC)\}}}\frac{d}{dx}\left[\text{arc cos}\sqrt{\left\{\frac{C(ax^2+c)}{c(Ax^2+C)}\right\}}\right]=\frac{1}{(Ax^2+C)\sqrt{(ax^2+c)}}.$$

10. Each of the functions

$$\frac{1}{\sqrt{(a^2-b^2)}}\text{arc cos}\left(\frac{a\cos x+b}{a+b\cos x}\right),\quad \frac{2}{\sqrt{(a^2-b^2)}}\text{arc tan}\left\{\sqrt{\left(\frac{a-b}{a+b}\right)}\tan\tfrac{1}{2}x\right\},$$

has the derivative $1/(a+b\cos x)$.

11. If $X=a+b\cos x+c\sin x$, and

$$y=\frac{1}{\sqrt{(a^2-b^2-c^2)}}\text{arc cos}\frac{aX-a^2+b^2+c^2}{X\sqrt{(b^2+c^2)}},$$

then $dy/dx=1/X$.

12. Find the equations of the tangent and normal at the point (x_0, y_0) of the circle $x^2+y^2=a^2$.

[Here $y=\sqrt{(a^2-x^2)}$, $dy/dx=-x/\sqrt{(a^2-x^2)}$, and the tangent is

$$y-y_0=(x-x_0)\{-x_0/\sqrt{(a^2-x_0{}^2)}\},$$

which may be put in the form $xx_0+yy_0=a^2$. The normal is $xy_0-yx_0=0$, which of course passes through the origin.]

13. Find the equations of the tangent and normal at any point of the ellipse $(x/a)^2+(y/b)^2=1$ and the hyperbola $(x/a)^2-(y/b)^2=1$.

14. The equations of the tangent and normal to the curve $x=\phi(t)$, $y=\psi(t)$, at the point (t), are

$$\frac{x-\phi(t)}{\phi'(t)}=\frac{y-\psi(t)}{\psi'(t)},\quad \{x-\phi(t)\}\phi'(t)+\{y-\psi(t)\}\psi'(t)=0.$$

15. Prove that the derivative of $F[f\{\phi(x)\}]$ is $F'[f\{\phi(x)\}]f'\{\phi(x)\}\phi'(x)$, and extend the result to still more complicated cases.

16. If u and v are functions of x, then

$$D_x\,\text{arc tan}\,(u/v)=(vD_xu-uD_xv)/(u^2+v^2).$$

17. If $\phi(x)=x^2\sin(1/x)$ when $x\neq 0$ and $\phi(0)=0$, then

$$\phi'(x)=2x\sin(1/x)-\cos(1/x)$$

when $x\neq 0$, and $\phi'(0)=0$. And $\phi'(x)$ is discontinuous for $x=0$ (cf. Ex. XLVIII. 44).

110. The Mean Value Theorem furnishes us with a proof of a result which is of essential importance for what follows:—*if $\phi'(x)=0$, throughout a certain interval of values of x, $\phi(x)$ is constant throughout that interval.*

For if a and b are any two values of x in the interval,

$$\phi(b)-\phi(a)=(b-a)\phi'\{a+\theta(b-a)\}=0.$$

An immediate corollary is that *if $\phi'(x)=\psi'(x)$, throughout a certain interval, the functions $\phi(x)$ and $\psi(x)$ differ, throughout that interval, by a constant.*

111. Integration. We have in this chapter seen how we can differentiate a given function $\phi(x)$—i.e. find its derivative—in a variety of cases, including all those of the commonest occurrence. It is natural to consider the converse question, that of determining *a function whose derivative is a given function.*

Suppose that $\psi(x)$ is the given function. Then we wish to determine a function such that $\phi'(x) = \psi(x)$. A little reflection shows us that this question may really be analysed into three parts.

(1) In the first place we want to know whether such a function as $\phi(x)$ *actually exists.* This is a purely theoretical question, and must be carefully distinguished from the practical question as to whether (supposing that there is such a function) we can find any simple formula to express it.

(2) We want to know whether it is possible that *more than one* such function should exist: i.e. we want to know whether our problem is one which admits of a *unique* solution or not; and if not, we want to know whether there is any simple relation between the different solutions which will enable us to express all of them in terms of any particular one.

(3) If there is a solution, we want to know *how to find an actual expression for it.*

It will throw light on the nature of these three distinct questions if we compare them with the three corresponding questions which arise with regard to the differentiation of functions.

(1) A function $\phi(x)$ may have a derivative for all values of x (like x^m, where m is a positive integer, or $\sin x$). It may generally, but not always have one (like $\sqrt[3]{x}$ or $\tan x$ or $\sec x$). Or again it may never have one: for example the function considered in Ex. XVII. 11, which is nowhere continuous, has obviously no derivative for any value of x. Of course, during this chapter, we have confined ourselves to functions which are continuous except for some special values of x. The example of the function $\sqrt[3]{x}$, however, shows that a continuous function may not have a derivative for some special value of x (in this case $x = 0$). Whether there are continuous functions which *never* have derivatives, or continuous curves which *never* have tangents, is a further question

which is at present beyond us. Common-sense says *No*: but, as we have already stated (§ 92 (2)), this is one of the cases in which higher mathematics has proved common-sense to be mistaken.

But at any rate it is clear enough that the theoretical question —has $\phi(x)$ a derivative $\phi'(x)$?—is one which has to be answered differently in different circumstances. And we may expect that the converse question—is there a function $\phi(x)$ of which $\psi(x)$ is the derivative?—will have different answers too. We have already seen that there are cases in which the answer is *No*: thus if $\psi(x)$ is the function which is equal to a, b, or c according as x is less than, equal to, or greater than 0, the answer is *No* (Exs. XLVIII. 45, XLIX. 3), unless $a=b=c$.

This is a case in which the given function is discontinuous. In what follows, however, we shall always suppose $\psi(x)$ continuous. And then the answer is, *Yes: if $\psi(x)$ is continuous there is always a function $\phi(x)$ such that $\phi'(x)=\psi(x)$.* To prove this would take us beyond our limits, however: in Ch. VII. we shall give a proof, not perfectly general, but general enough to deal with the simplest and most interesting cases that arise.

(2) The second question presents no difficulties. In the case of differentiation we have a direct definition of the derivative which makes it clear from the beginning that there cannot possibly be more than one. In the case of the converse problem the answer is almost equally simple. It is that if $\phi(x)$ is one solution of the problem $\phi(x)+C$ is another, for any value of the constant C: and that all possible solutions are comprised in the form $\phi(x)+C$. This follows at once from § 110.

(3) The practical problem of actually finding $\phi'(x)$ is as a rule a fairly simple one. We have already shown how it can be done in a number of cases, and the theorem of § 108, in conjunction with the rules of § 94, make the problem easy enough in the case of any function defined by some finite combination of the ordinary functional symbols. The converse problem is much more difficult. The nature of the difficulties will appear more clearly later on.

DEFINITIONS. *If $\psi(x)$ is the derivative of $\phi(x)$, we call $\phi(x)$ the* **integral** *or* **integral function** *of $\psi(x)$. The operation of forming $\psi(x)$ from $\phi(x)$ we call* **integration**.

We shall use the notation

$$\phi(x) = \int \psi(x)\, dx.$$

It is hardly necessary to point out that $\int .. dx$ like d/dx must, at present at any rate, be regarded purely as a *symbol of operation*: the $\int$ and the dx no more mean anything when taken by themselves than do the d and dx of the other operative symbol d/dx. The reason for this notation will be explained in Ch. VII.

112. The practical problem of integration. The results of the earlier part of this chapter enable us to write down at once the integrals of some of the commonest functions. Thus

$$\int x^m dx = \frac{x^{m+1}}{m+1}, \quad \int \cos x\, dx = \sin x, \quad \int \sin x\, dx = -\cos x \ldots (1).$$

These formulae must be understood as meaning that the function on the right-hand side is *one* integral of that under the sign of integration. The *most general* integral is of course obtained by adding to the former a constant C, known as the **arbitrary constant** of integration.

There is however one case of exception to the first formula, that in which $m = -1$. In this case the formula becomes nugatory, as is only to be expected, since we have already (Ex. XLIV. 5) seen that $1/x$ cannot be the derivative of any polynomial or rational fraction. And in fact it can be proved (though the proof is too detailed and tedious to be inserted here) that it is impossible to form, by means of a finite combination of the functional signs which correspond to any of the classes of functions which we have so far considered—signs such as $+$, $\times$, $\div$, $\sqrt{}$, sin, arc sin,—a function of x whose derivative is $1/x$. Some further discussion of this point will be found in Ch. IX. For the present we shall be content to assume that, if there is such a function, it is an *essentially new* function.

That there really is a function $F(x)$ such that $D_x F(x) = 1/x$ will be proved in the next chapter; and the properties of this function will be investigated in Ch. IX. For the present we shall simply assume the existence of such a function, and we shall call it **the logarithmic function** and denote it by **log** x; so that

$$\int \frac{dx}{x} = \log x \quad \text{.........................(2).}$$

The four formulae (1) and (2) are the four most fundamental *standard forms* of the Integral Calculus. To them should perhaps be added two more, viz.

$$\int \frac{dx}{1+x^2} = \text{arc} \tan x, \quad \int \frac{dx}{\sqrt{(1-x^2)}} = \pm \text{arc} \sin x^* \quad \text{...(3).}$$

113. Polynomials. All the general theorems of § 94 may of course be also stated as theorems in integration. Thus we have, to begin with, the formulae

$$\int \{f(x) + F(x)\}\, dx = \int f(x)\, dx + \int F(x)\, dx \quad \text{.........(1),}$$

$$\int \kappa f(x)\, dx = \kappa \int f(x)\, dx \quad \text{.....................(2).}$$

Here it is assumed, of course, that the arbitrary constants are adjusted properly. Thus the formula (1) asserts that the sum of *any* integral of $f(x)$ and *any* integral of $F(x)$ is *an* integral of $f(x) + F(x)$.

These theorems enable us to write down at once the integral of any function of the form $\Sigma A_\nu f_\nu(x)$, the sum of a finite number of constant multiples of functions whose integrals are known. In particular we can write down the integral of any *polynomial*: in fact

$$\int (a_0 x^n + a_1 x^{n-1} + \dots + a_n)\, dx = \frac{a_0 x^{n+1}}{n+1} + \frac{a_1 x^n}{n} + \dots + a_n x.$$

114. Rational Functions. After integrating polynomials it is natural to turn our attention next to *rational functions.* Let us suppose $R(x)$ to be any rational function expressed in the standard form of § 98, viz. as the sum of a polynomial $\Pi(x)$ and a number of terms of the form $A/(x-\alpha)^p$.

We can at once write down the integrals of the polynomial and of all the other terms except those for which $p = 1$, since

$$\int \frac{A}{(x-\alpha)^p}\, dx = -\frac{A}{p-1} \frac{1}{(x-\alpha)^{p-1}},$$

whether α be real or complex (§ 98).

* See § 100 for the rule for determining the ambiguous sign.

The terms for which $p=1$ present rather more difficulty. And it is convenient at this stage to introduce another general theorem in integration.

It follows immediately from Theorem (8) of § 94 that if $\int \psi(x)\,dx = \phi(x)$, and a and b are real, then

$$\int \psi(ax+b)\,dx = (1/a)\,\phi(ax+b) \quad \text{...........(3)}.$$

Thus, for example,

$$\int \frac{dx}{ax+b} = (1/a)\log(ax+b),$$

and, in particular, if α is real,

$$\int \frac{dx}{x-\alpha} = \log(x-\alpha).$$

We can therefore write down the integrals of all the terms in $R(x)$ for which $p=1$ and α is real. There remain the terms for which $p=1$ and α is complex.

In order to deal with these we shall introduce a restrictive hypothesis, viz. that $R(x)$ is a real function—i.e. that all its coefficients are real. Then if $\alpha=\gamma+i\delta$ is a root of $Q(x)=0$, m times repeated, so is $\alpha'=\gamma-i\delta$. Moreover, if the partial fractions corresponding to the factor $(x-\alpha)^m$ are $\Sigma A_p/(x-\alpha)^p$, those corresponding to the factor $(x-\alpha')^m$ are $\Sigma A_p'/(x-\alpha')^p$, where A_p' is conjugate (Ch. III, § 30) to A_p. This follows from the nature of the algebraical processes by means of which the partial fractions can be found, and which are explained at length in treatises on Algebra*.

Thus if a term $(\lambda+i\mu)/(x-\gamma-i\delta)$ occurs in the standard form of $R(x)$, so will a term $(\lambda-i\mu)/(x-\gamma+i\delta)$; and the sum of these two terms is

$$2\{\lambda(x-\gamma)-\mu\delta\}/\{(x-\gamma)^2+\delta^2\}.$$

This fraction is in reality the most general fraction of the form

$$(Ax+B)/(ax^2+2bx+c),$$

where $b^2<ac$. The reader will easily verify the equivalence of

* See, for example, Chrystal's *Algebra*, vol. I, pp. 151–9.

the two forms, the formulae which express λ, μ, γ, δ in terms of A, B, a, b, c being

$$\lambda = A/2a, \quad \mu = -D/(2a\sqrt{\Delta}), \quad \gamma = -b/a, \quad \delta = \sqrt{\Delta}/a,$$

where $\Delta = ac - b^2$, and $D = aB - bA$.

We shall now introduce another general theorem in integration, which follows at once from the theorem of § 108: viz.

$$\int F'\{f(x)\} f'(x)\, dx = F\{f(x)\} \quad \ldots\ldots\ldots\ldots\ldots(4).$$

If in particular we suppose $F\{f(x)\}$ to be $\log f(x)$, so that $F'\{f(x)\} = 1/f(x)$, we obtain

$$\int \frac{f'(x)}{f(x)}\, dx = \log f(x);$$

and if we further suppose that $f(x) = (x-\lambda)^2 + \mu^2$ we obtain

$$\int \frac{2(x-\lambda)}{(x-\lambda)^2 + \mu^2}\, dx = \log\{(x-\lambda)^2 + \mu^2\}.$$

Again, in virtue of the equations (3) of § 112 and (3) above

$$\int \frac{-2\delta\mu}{(x-\lambda)^2 + \mu^2}\, dx = -2\delta \arctan\left(\frac{x-\lambda}{\mu}\right).$$

These two formulae enable us to integrate the sum of the two terms which we have been considering in the expression of $R(x)$; and we are thus enabled to write down the integral of any rational function, if all the factors of its denominator can be determined. The integral of any such function is composed of *the sum of a polynomial, a number of rational fractions of the type*

$$-A/\{(p-1)(x-\alpha)^{p-1}\},$$

a number of logarithmic functions, and a number of inverse tangents. It only remains to add that if α is complex such a fraction as

$$-A/\{(p-1)(x-\alpha)^{p-1}\}$$

always occurs in conjunction with another in which A and α are replaced by the complex numbers conjugate to them, and that the sum of the two fractions is a real rational fraction.

Examples LI. 1. The integral of the function $(Ax+B)/(ax^2+2bx+c)$ may be expressed in the form

$$\frac{A}{2a}\log X + \frac{D}{2a\sqrt{(-\Delta)}}[\log\{ax+b-\sqrt{(-\Delta)}\} - \log\{ax+b+\sqrt{(-\Delta)}\}]$$

(where $X = ax^2+2bx+c$) if $\Delta < 0$, and in the form

$$\frac{A}{2a}\log X + \frac{D}{a\sqrt{\Delta}}\arctan\left(\frac{ax+b}{\sqrt{\Delta}}\right)$$

if $\Delta > 0$.

2. In the particular case in which $ac=b^2$ the integral is

$$-D/\{a(ax+b)\}+(A/a)\log\{x+(b/a)\}.$$

3. Show that if the roots of $Q(x)=0$ are all real and distinct, and $P(x)$ is of lower degree than $Q(x)$, then

$$\int R(x)\,dx=\sum\frac{P(\alpha)}{Q'(\alpha)}\log(x-\alpha)$$

the summation applying to all the roots α of $Q(x)=0$.

[That the fraction corresponding to α is $\dfrac{P(\alpha)}{Q'(\alpha)}\dfrac{1}{x-\alpha}$ follows from the fact that $Q(x)/(x-\alpha)\to Q'(\alpha)$ and so $(x-\alpha)R(x)\to P(\alpha)/Q'(\alpha)$, as $x\to\alpha$.]

4. If all the roots of $Q(x)$ are real and one (α) is a double root, the rest being simple roots, and $P(x)$ is of lower degree than $Q(x)$, then the integral is $A/(x-\alpha)+A'\log(x-\alpha)+\sum B\log(x-\beta)$, where

$$A=-2P(\alpha)/Q''(\alpha),\quad A'=\tfrac{2}{3}\{3P'(\alpha)Q''(\alpha)-P(\alpha)Q'''(\alpha)\}/\{Q''(\alpha)\}^2,\quad B=P(\beta)/Q'(\beta),$$

and the summation applies to all roots β of $Q(x)=0$ other than α.

5. Find
$$\int\frac{dx}{\{(x-1)(x^2+1)\}^2}.$$

[The expression in partial fractions is

$$\frac{1}{4(x-1)^2}-\frac{1}{2(x-1)}-\frac{i}{8(x-i)^2}+\frac{2-i}{8(x-i)}+\frac{i}{8(x+i)^2}+\frac{2+i}{8(x+i)},$$

and the integral is

$$-\frac{1}{4(x-1)}-\frac{1}{4(x^2+1)}-\tfrac{1}{2}\log(x-1)+\tfrac{1}{4}\log(x^2+1)+\tfrac{1}{4}\arctan x^*.]$$

6. Integrate

$$x/\{(x-a)(x-b)(x-c)\},\quad x/\{(x-a)^2(x-b)\},\quad x/(x-a)^3,$$
$$x/\{(x^2+a^2)(x^2+b^2)\},\quad x^2/\{(x^2+a^2)(x^2+b^2)\},\quad x^3/\{(x^2+a^2)(x^2+b^2)\},$$
$$(x+1)/\{x^2(x-1)\},\quad (x+1)/\{x(x-1)^2\},\quad (x+1)/\{x(x-1)\}^2,$$
$$(x^2-1)/\{x^2(x^2+1)\},\quad (x^2-1)/\{x(x^2+1)^2\},\quad (x^2-1)/\{x(x^2+1)\}^2.$$

7. Prove the formulae:

$$\int\frac{dx}{1+x^4}=\frac{1}{4\sqrt{2}}\left\{\log(1+x\sqrt{2}+x^2)-\log(1-x\sqrt{2}+x^2)+2\arctan\left(\frac{x\sqrt{2}}{1-x^2}\right)\right\},$$

$$\int\frac{x^2\,dx}{1+x^4}=\frac{1}{4\sqrt{2}}\left\{\log(1-x\sqrt{2}+x^2)-\log(1+x\sqrt{2}+x^2)+2\arctan\left(\frac{x\sqrt{2}}{1-x^2}\right)\right\},$$

$$\int\frac{dx}{1+x^2+x^4}=\frac{1}{4\sqrt{3}}\left\{\sqrt{3}[\log(1+x+x^2)-\log(1-x+x^2)]+2\arctan\left(\frac{x\sqrt{3}}{1-x^2}\right)\right\}.$$

* In this case the application of the general method of § 114 is fairly simple. In more complicated cases the labour involved is sometimes prohibitive, and other devices have to be used. We have, moreover, assumed that all the factors of the denominator can be determined. If this is not the case the method of partial fractions fails, and recourse must be had to other methods. For further information concerning the integration of rational functions the reader may be referred to Goursat's *Cours d'Analyse*, t. i, pp. 234 *et seq.*, and to the author's tract *The integration of functions of a single variable*, pp. 10 *et seq.*

115. Algebraical Functions. We naturally pass on next to the question of the integration of *algebraical* functions. We shall confine our attention to *explicit* algebraical functions (Ch. II, § 16).

We have to consider the problem of integrating y, where y is an explicit algebraical function of x. It is however convenient to consider an apparently more general integral, viz.

$$\int R(x, y)\, dx,$$

where $R(x, y)$ is any rational function of x and y. The greater generality of this form is only apparent, since (Ex. XV. 6) the function $R(x, y)$ is itself an algebraical function of x. The choice of this form is in fact dictated simply by motives of convenience: such a function as

$$\{x + \sqrt{(ax^2 + 2bx + c)}\}/\{x - \sqrt{(ax^2 + 2bx + c)}\}$$

is far more conveniently regarded as a rational function of x and the simple algebraical function $\sqrt{(ax^2 + 2bx + c)}$, than directly as itself an algebraical function of x.

116. Integration by substitution and rationalisation. It follows from equation (4) of § 114 that if $\int \psi(x)\, dx = \phi(x)$, then

$$\int \psi\{f(t)\} f'(t)\, dt = \phi\{f(t)\} \qquad \text{..............(1).}$$

This equation supplies us with a method for determining the integral of $\psi(x)$ in a large number of cases in which the form of the integral is not directly obvious. It may be stated as a rule as follows: *put $x = f(t)$, where $f(t)$ is any function of a new variable t which it may be convenient to choose; multiply by $f'(t)$ and determine (if possible) the integral of $\psi\{f(t)\} f'(t)$; express the result in terms of x.* It will often be found that the function of t to which we are led by the application of this rule is one whose integral can easily be calculated. This is always so, for example, if it is a rational function, and it is very often possible to choose the relation between x and t so that this shall be the case. Thus the integral of $R(\sqrt{x})$, where R denotes a rational function, is reduced by the substitution $x = t^2$ to the integral of $2tR(t^2)$, i.e. to the integral of a rational function of t. This method of integration is called **integration by rationalisation,** and is of extremely wide application.

Its application to the problem immediately under consideration is obvious. *If we can find a variable t such that x and y are both rational functions of t, say* $x = R_1(t)$, $y = R_2(t)$, *then*

$$\int R(x, y)\,dx = \int R\{R_1(t), R_2(t)\}\, R_1'(t)\,dt,$$

and the latter integral, being that of a rational function of t, can be calculated by the methods of § 114.

It would carry us beyond our present range to enter upon any general discussion as to when it is and when it is not possible to find an auxiliary variable t connected with x and y in the manner indicated above. We shall only consider a few simple and interesting special cases.

117. Integrals connected with conics. Let us suppose that x and y are connected by an equation of the form

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0;$$

in other words that the graph of y, considered as a function of x, is a conic. Suppose that (ξ, η) is any point on the conic, and let $x - \xi = X$, $y - \eta = Y$. If the relation between x and y is expressed in terms of X and Y it assumes the form

$$aX^2 + 2hXY + bY^2 + 2GX + 2FY = 0,$$

where $F = h\xi + b\eta + f$, $G = a\xi + h\eta + g$. In this equation put $Y = tX$. It will then be found that X and Y can both be expressed as rational functions of t, and therefore x and y can be so expressed, the actual formulae being

$$x - \xi = -2(G + Ft)/(a + 2ht + bt^2), \quad y - \eta = -2t(G + Ft)/(a + 2ht + bt^2).$$

Hence the process of rationalisation described in the last section can be carried out.

The reader should verify that

$$hx + by + f = -\tfrac{1}{2}(a + 2ht + bt^2)\frac{dx}{dt},$$

so that
$$\int \frac{dx}{hx + by + f} = -2\int \frac{dt}{a + 2ht + bt^2},$$

a formula which will be useful later on.

118. The integral $\int R\{x, \sqrt{(ax^2+2bx+c)}\}\,dx$. The most important case is that in which the relation between x and y is

$$y^2 = ax^2 + 2bx + c.$$

Let ξ, $\sqrt{(a\xi^2+2b\xi+c)}$ be the coordinates of any point on the conic. The relation between X and Y is

$$aX^2 - Y^2 + 2(a\xi + b)X - 2\eta Y = 0,$$

and the formulae expressing x and y in terms of t are

$$x - \xi = -\frac{2(a\xi + b - t\eta)}{a - t^2}, \quad y - \eta = -\frac{2t(a\xi + b - t\eta)}{a - t^2} \quad \ldots(1).$$

Consider, for example, the integral $\int \frac{dx}{y}$. If $\Delta = ac - b^2$, there are three cases to consider, viz. those in which

(i) $a>0$, $\Delta>0$; (ii) $a>0$, $\Delta<0$; (iii) $a<0$, $\Delta<0$.

If $a<0$ and $\Delta>0$, $ax^2+2bx+c$ is always negative, so that this case is not of any interest.

If $a>0$, $\Delta>0$, the conic is a hyperbola with one branch entirely above the axis of x, which is described once as t varies from $-\sqrt{a}$ to $+\sqrt{a}$. If (ξ, η) lies on this branch, $\eta>0$ and $-\sqrt{a}<t<\sqrt{a}$. Since now $h=f=0$, $b=-1$, the last formula of § 117 gives

$$\int \frac{dx}{y} = 2\int \frac{dt}{a-t^2} = \frac{1}{\sqrt{a}}\{\log(\sqrt{a}+t) - \log(\sqrt{a}-t)\}.$$

But it follows from the equations (1) that

$$(x-\xi)\sqrt{a} \pm (y-\eta) = -2(a\xi + b - t\eta)/(\sqrt{a} \mp t)$$

and so*

$$\int \frac{dx}{y} = \frac{1}{\sqrt{a}} \log\left(\frac{\sqrt{a}+t}{\sqrt{a}-t}\right) = \frac{1}{2\sqrt{a}} \log\left\{\frac{(x-\xi)\sqrt{a}+(y-\eta)}{(x-\xi)\sqrt{a}-(y-\eta)}\right\}^2 \quad \ldots\ldots\ldots(2),$$

the logarithm being written in this form in order to avoid any possible difficulty as to the sign of the function inside the large bracket. This equation is true for all pairs of values of ξ and η related as above. A particularly simple form of the integral may be found as follows. Since we may add any constant to the right-hand side, we have

$$\int \frac{dx}{y} = \frac{1}{2\sqrt{a}} \log\{(x-\xi)\sqrt{a}+(y-\eta)\}^2 + \frac{1}{2\sqrt{a}} \log\left\{\frac{\eta - \xi\sqrt{a}}{(x-\xi)\sqrt{a}-(y-\eta)}\right\}^2.$$

Now suppose that $\xi \to -\infty$. Then it is easy to see that $\eta \to +\infty$, $\xi\sqrt{a}+\eta \to -b/\sqrt{a}$, and that the contents of the last large bracket tend to

* In the succeeding discussion we anticipate the fundamental properties of the logarithm, which will be proved later on, viz. that $\log u$ is continuous for all positive values of u, $\log 1 = 0$, $\log(1/u) = -\log u$, $\log uv = \log u + \log v$.

unity. We thus obtain the formula

$$\int \frac{dx}{y} = \frac{1}{2\sqrt{a}} \log \left(x\sqrt{a} + y + \frac{b}{\sqrt{a}} \right)^2 \quad \text{.....................(3).}$$

If in particular $a=1$, $b=0$, $c=a^2$ we obtain

$$\int \frac{dt}{\sqrt{(x^2+a^2)}} = \log\{x + \sqrt{(x^2+a^2)}\} \quad \text{.....................(4).}$$

The truth of this equation may be at once verified by differentiation of the right-hand side. If we transform the integral by the substitution $x^2+a^2=u^2$, $u=\sqrt{(x^2-a^2)}$ we obtain, on writing x again for u,

$$\int \frac{dx}{\sqrt{(x^2-a^2)}} = \log\{x + \sqrt{(x^2-a^2)}\} \quad \text{.....................(5).}$$

This integral may also be calculated directly by an argument similar to that used for the integral (4). It is the simplest example of case (ii) above. The reader should associate with these two formulae the third formula

$$\int \frac{dx}{\sqrt{(a^2-x^2)}} = \arcsin(x/a) \quad \text{...........................(6).}$$

This integral corresponds to the case (iii) above. The formula appears very different from (4) and (5): the reader will hardly be in a position to appreciate the connection between them until he has read Chs. IX and X. In the last formula it is supposed that a is positive: if a is negative the integral function is $\arcsin(x/|a|) = -\arcsin(x/a)$ (cf. § 100).

119. The integral $\int \frac{\lambda x + \mu}{\sqrt{(ax^2 + 2bx + c)}}\, dx$. This integral can be integrated in all cases by means of the results of the preceding sections. It is most convenient to proceed as follows. Since

$$\lambda x + \mu \equiv (\lambda/a)(ax + b) + \mu - (\lambda b/a),$$

$$\int \frac{ax + b}{\sqrt{(ax^2 + 2bx + c)}}\, dx = \sqrt{(ax^2 + 2bx + c)},$$

we have

$$\int \frac{(\lambda x + \mu)\, dx}{\sqrt{(ax^2 + 2bx + c)}} = (\lambda/a)\sqrt{(ax^2 + 2bx + c)} + \gamma \int \frac{dx}{\sqrt{(ax^2 + 2bx + c)}},$$

where $\gamma = \mu - (\lambda b/a)$. In the last integral a may be positive or negative. If it is positive we put $x\sqrt{a} + (b/\sqrt{a}) = t$, when we obtain

$$\frac{1}{\sqrt{a}} \int \frac{dt}{\sqrt{(t^2 + \kappa)}}$$

where $\kappa = (ac - b^2)/a$. If a is negative we write A for $-a$ and put $x\sqrt{A} - (b/\sqrt{A}) = t$, when we obtain

$$\frac{1}{\sqrt{(-a)}} \int \frac{dt}{\sqrt{(-\kappa - t^2)}}.$$

It thus appears that in any case the calculation of the integral may be made to depend on that considered in § 118, viz. one or other of the three integrals

$$\int \frac{dt}{\sqrt{(t^2+\alpha^2)}}, \quad \int \frac{dt}{\sqrt{(t^2-\alpha^2)}}, \quad \int \frac{dt}{\sqrt{(\alpha^2-t^2)}}.$$

120. The integral $\int (\lambda x+\mu)\sqrt{(ax^2+2bx+c)}\,dx$. In exactly the same way we find

$$\int (\lambda x+\mu)\sqrt{(ax^2+2bx+c)}\,dx = (\lambda/3a)(ax^2+2bx+c)^{3/2}+\gamma\int\sqrt{(ax^2+2bx+c)}\,dx;$$

and the last integral may be reduced to one or other of the three forms

$$\int\sqrt{(t^2+a^2)}\,dt, \quad \int\sqrt{(t^2-a^2)}\,dt, \quad \int\sqrt{(a^2-t^2)}\,dt.$$

In order to obtain these integrals it is convenient to introduce at this point another general theorem in integration.

121. Integration by parts. The theorem of *integration by parts* is merely another way of stating the rule for the differentiation of a product (§ 94). It follows at once from Theorem (3) of § 94 that

$$\int f'(x)\,F(x)\,dx = f(x)\,F(x) - \int f(x)\,F'(x)\,dx.$$

It may happen that the function which we wish to integrate is expressible in the form $f'(x)\,F(x)$, and that $f(x)\,F'(x)$ can be integrated. Thus suppose that $\phi(x) = x\psi(x)$, where $\psi(x)$ is the second derivative of a known function $\chi(x)$. Then

$$\int \phi(x)dx = \int x\chi''(x)dx = x\chi'(x) - \int \chi'(x)dx = x\chi'(x) - \chi(x).$$

We can illustrate the working of this method of integration by applying it to the integrals of the last section. Taking

$$f(x) = ax+b, \quad F(x) = \sqrt{(ax^2+2bx+c)} = y,$$

we obtain

$$a\int y\,dx = (ax+b)y - \int \frac{(ax+b)^2}{y}\,dx = (ax+b)y - a\int y\,dx + (ac-b^2)\int\frac{dx}{y},$$

so that

$$\int y\,dx = \frac{ax+b}{2a}\,y + \frac{ac-b^2}{2a}\int\frac{dx}{y};$$

and we have already seen (§ 118) how to determine the last integral.

Examples LII. 1. Prove that, if $a>0$,

$$\int \sqrt{(x^2+a^2)}\,dx = \tfrac{1}{2}x\sqrt{(x^2+a^2)} + \tfrac{1}{2}a^2 \log\{x+\sqrt{(x^2+a^2)}\},$$

$$\int \sqrt{(x^2-a^2)}\,dx = \tfrac{1}{2}x\sqrt{(x^2-a^2)} - \tfrac{1}{2}a^2 \log\{x+\sqrt{(x^2-a^2)}\},$$

$$\int \sqrt{(a^2-x^2)}\,dx = \tfrac{1}{2}x\sqrt{(a^2-x^2)} + \tfrac{1}{2}a^2 \arcsin(x/a).$$

2. Calculate the integrals $\int \frac{dx}{\sqrt{(a^2-x^2)}}$, $\int \sqrt{(a^2-x^2)}\,dx$ by means of the substitution $x = a\sin\theta$, and verify that the results agree with those obtained in § 118 and Ex. 1.

3. Calculate $\int x(x+a)^m\,dx$, m being any rational number, in three ways, (i) by integration by parts, (ii) by the substitution $(x+a)^m = t$, (iii) by writing $(x+a)-a$ for x; and verify that the results agree.

4. Prove, by means of the substitutions $ax+b=1/t$ and $x=1/u$, that (in the notation of § 118)

$$\int \frac{dx}{y^3} = \frac{ax+b}{\Delta y}, \quad \int \frac{x\,dx}{y^3} = -\frac{bx+c}{\Delta y}.$$

5. Integrate

$1/\{(1+x)\sqrt{x}\}$, $\sqrt{x}/(1+x)$, $1/\{x\sqrt{(1+x)}\}$, $x/\sqrt{(1+x)}$, $1/\sqrt{\{x(1+x)\}}$,
$1/\sqrt{\{x(x-1)\}}$, $1/\sqrt{\{x(1-x)\}}$, $\sqrt{\{(1+x)/(1-x)\}}$, $x\sqrt{(a+bx)}$, $x^2/\sqrt{(a+bx)}$,
$1/\{x\sqrt{(x^2+a^2)}\}$, $1/\{x^2\sqrt{(x^2-a^2)}\}$, $1/\{x^3\sqrt{(a^2-x^2)}\}$, $x^3/\sqrt{(a^2-x^2)}$.

6. Integrate $1/\sqrt{\{(x-a)(b-x)\}}$ in three ways, (i) by the methods of the preceding sections, (ii) by the substitution $(b-x)/(x-a)=t^2$, (iii) by the substitution $x = a\cos^2\theta + b\sin^2\theta$; and verify that the results agree.

7. Integrate $\sqrt{\{(x-a)(b-x)\}}$ and $\sqrt{\{(b-x)/(x-a)\}}$.

8. Show, by means of the substitution $2x+a+b=\tfrac{1}{2}(a-b)\{t^2+(1/t^2)\}$, or by multiplying numerator and denominator by $\sqrt{(x+a)}-\sqrt{(x+b)}$, that, if $a>b$,

$$\int \frac{dx}{\sqrt{(x+a)}+\sqrt{(x+b)}} = \tfrac{1}{2}\sqrt{(a-b)}\{t+(1/3t^3)\}.$$

9. Find a substitution which will reduce $\int \frac{dx}{(x+a)^{3/2}+(x-a)^{3/2}}$ to the integral of a rational function. (*Math. Trip.* 1899.)

10. Show that $\int R\{x, \sqrt[n]{(ax+b)}\}\,dx$ is reduced, by the substitution $ax+b=y^n$, to the integral of a rational function.

11. Prove that

$$\int f''(x)F(x)\,dx = f'(x)F(x) - f(x)F'(x) + \int f(x)F''(x)\,dx,$$

and generally

$$\int f^{(n)}(x)F(x)\,dx = f^{(n-1)}(x)F(x) - f^{(n-2)}(x)F'(x) + \dots + (-1)^n \int f(x)F^{(n)}(x)\,dx.$$

12. Hence integrate $x^n\phi^{(n+1)}(x)$, i.e. $x^n\psi(x)$, where $\psi(x)$ is a function which can be integrated $n+1$ times. In particular integrate $x^n(ax+b)^p$, n being a positive integer, and p any rational number other than -1.

13. The integral $\int(1+x)^p x^q\,dx$, where p and q are rational, can be found in three cases, viz. (i) if p is an integer, (ii) if q is an integer, (iii) if $p+q$ is an integer.

[In case (i) put $x=u^s$, where s is the denominator of q: in case (ii) put $1+x=t^s$, where s is the denominator of p: in case (iii) put $1+x=xt^s$, where s is the denominator of p.]

14. The integral $\int x^m(ax^n+b)^q\,dx$ can be reduced to the preceding integral by the substitution $ax^n=bt$.

[In practice it is often most convenient to calculate a particular integral of this kind by a *formula of reduction* (v. Misc. Ex. 40).]

15. The integral $\int R\{x, \sqrt{(ax+b)}, \sqrt{(cx+d)}\}\,dx$ can be reduced to that of a rational function by the substitution

$$4x=-(b/a)\{t+(1/t)\}^2-(d/c)\{t-(1/t)\}^2.$$

16. Show how to calculate the integral $\int\frac{dx}{(x-p)\sqrt{(ax^2+2bx+c)}}$ by means of the substitution $x-p=1/t$.

17. Show by means of the substitution $y=\sqrt{(ax^2+2bx+c)}/(x-p)$ that

$$\int\frac{dx}{(x-p)\sqrt{(ax^2+2bx+c)}}=\int\frac{dy}{\sqrt{\{\lambda y^2-\mu\}}},$$

where $\lambda=ap^2+2bp+c$, $\mu=ac-b^2$; and hence evaluate the integral.

18. Calculate the integrals of

$$1/\{(x-1)\sqrt{(x^2+1)}\},\quad 1/\{(x+1)\sqrt{(1+2x-x^2)}\}$$

by means of each of the preceding methods, and verify the agreement of the results*.

19. Reduce $\int R(x, y)\,dx$, where $y^2(x-y)=x^2$, to the integral of a rational function. [Putting $y=tx$ we obtain $x=1/\{t^2(1-t)\}$, $y=1/\{t(1-t)\}$.]

20. Reduce the integral in the same way when (*a*) $y(x-y)^2=x$, (*b*) $(x^2+y^2)^2=a^2(x^2-y^2)$. [In case (*a*) put $x-y=t$: in case (*b*) put $x^2+y^2=t(x-y)$, when we obtain $x=a^2t(t^2+a^2)/(t^4+a^4)$, $y=a^2t(t^2-a^2)/(t^4+a^4)$]

21. If $y(x-y)^2=x$ then $\int\frac{dx}{x-3y}=\frac{1}{2}\log\{(x-y)^2-1\}$.

22. If $(x^2+y^2)^2=2c^2(x^2-y^2)$ then $\int\frac{dx}{y(x^2+y^2+c^2)}=-\frac{1}{c^2}\log\left(\frac{x^2+y^2}{x-y}\right)$.

* See also Misc. Exs. 33 *et seq.*

122. Transcendental Functions. Owing to the immense variety of the different classes of transcendental functions, the theory of their integration is a good deal less systematic than that of the integration of rational or algebraical functions. We shall consider in order a few classes of transcendental functions whose integrals can always be found.

123. Polynomials in cosines and sines of multiples of x. We can always integrate any function which is the sum of a finite number of terms such as

$$A\,(\cos ax)^m(\sin ax)^{m'}(\cos bx)^n(\sin bx)^{n'}\ldots$$

where m, m', n, n', ... are positive integers and a, b, ... any real numbers whatever. For such a term can be expressed as the sum of a finite number of terms of the types

$$\alpha\cos\{(pa+qb+\ldots)x\},\quad \beta\sin\{(p'a+q'b+\ldots)x\}$$

and the integrals of these terms may be at once written down.

Examples LIII. 1. Integrate $\sin^3 x\cos^2 2x$. In this case we use the formulae

$$\sin^3 x=\tfrac{1}{4}(3\sin x-\sin 3x),\quad \cos^2 2x=\tfrac{1}{2}(1+\cos 4x).$$

Multiplying these two expressions and replacing $\sin x\cos 4x$, for example, by $\frac{1}{2}(\sin 5x-\sin 3x)$, we obtain

$$\tfrac{1}{16}\int(7\sin x-5\sin 3x+3\sin 5x-\sin 7x)\,dx$$
$$=-\tfrac{7}{16}\cos x+\tfrac{5}{48}\cos 3x-\tfrac{3}{80}\cos 5x+\tfrac{1}{112}\cos 7x.$$

The integral may of course be obtained in a different form by different methods. For example

$$\int\sin^3 x\cos^2 2x\,dx=\int(4\cos^4 x-4\cos^2 x+1)(1-\cos^2 x)\sin x\,dx,$$

which reduces, on making the substitution $\cos x=t$, to

$$\int(4t^6-8t^4+5t^2-1)\,dt=\tfrac{4}{7}\cos^7 x-\tfrac{8}{5}\cos^5 x+\tfrac{5}{3}\cos^3 x-\cos x.$$

It may of course be verified that this expression and the integral already obtained differ only by a constant.

2. Integrate by any method $\cos ax\cos bx$, $\sin ax\sin bx$, $\cos ax\sin bx$, $\cos^2 x$, $\sin^3 x$, $\cos^4 x$, $\cos x\cos 2x\cos 3x$, $\cos\frac{1}{2}x\cos 2x$, $\cos^3 2x\sin^2 3x$, $\cos^5 x\sin^7 x$. [In cases of this kind it is also sometimes convenient to use a formula of reduction (Misc. Ex. 40).]

124. The integrals $\int x^n\cos x\,dx$, $\int x^n\sin x\,dx$ and associated integrals. The method of integration by parts enables us to generalise the preceding results. For

$$\int x^n \cos x\,dx = \quad x^n \sin x - n\int x^{n-1} \sin x\,dx,$$

$$\int x^n \sin x\,dx = -x^n \cos x + n\int x^{n-1} \cos x\,dx,$$

and clearly the integrals can be calculated completely by a repetition of this process. It follows that we can always calculate $\int x^n \cos ax\,dx$, $\int x^n \sin ax\,dx$; and so by a process similar to that of the preceding paragraph, we can calculate

$$\int P(x, \cos ax, \sin ax, \cos bx, \sin bx, \ldots)\,dx,$$

where P is any polynomial.

Examples LIV. 1. Integrate $x\sin x$, $x^2\cos x$, $(x\cos x)^2$, $(x\sin x\sin 2x)^2$, $x\sin^2 x\cos^4 x$, $(x\sin\frac{1}{3}x)^3$.

2. Find polynomials P and Q such that

$$\int[(3x-1)\cos x+(1-2x)\sin x]\,dx=P\cos x+Q\sin x.$$

3. Prove that $\int x^n \cos x\,dx=P_n\cos x+Q_n\sin x$, where

$$P_n=nx^{n-1}-n(n-1)(n-2)x^{n-3}+\ldots,\quad Q_n=x^n-n(n-1)x^{n-2}+\ldots.$$

125. Rational Functions of $\cos x$ **and** $\sin x$. The integral of any rational function of $\cos x$ and $\sin x$ may be calculated by the substitution $\tan\frac{1}{2}x=t$. For

$$\cos x=\frac{1-t^2}{1+t^2},\quad \sin x=\frac{2t}{1+t^2},\quad \frac{dx}{dt}=\frac{2}{1+t^2},$$

so that the substitution reduces the integral to that of a rational function of t.

Examples LV. 1. Prove that

$$\int\sec x\,dx=\log(\sec x+\tan x),\quad \int\operatorname{cosec} x\,dx=\log\tan\tfrac{1}{2}x.$$

[Another form of the first integral is $\log\tan(\frac{1}{4}\pi+\frac{1}{2}x)$; a third form is $\frac{1}{2}\log\{(1+\sin x)/(1-\sin x)\}$.]

2. $\int\tan x\,dx=-\log\cos x$, $\int\cot x\,dx=\log\sin x$, $\int\sec^2 x\,dx=\tan x$, $\int\operatorname{cosec}^2 x\,dx=-\cot x$, $\int\tan x\sec x\,dx=\sec x$, $\int\cot x\operatorname{cosec} x\,dx=-\operatorname{cosec} x$.

[These integrals are included in the general form, but there is no need to use a substitution, as the results follow at once from § 100.]

3. Show that the integral of $1/(a+b\cos x)$, where $a+b$ is positive, may be expressed in one or other of the forms

$$\frac{2}{\sqrt{(a^2-b^2)}}\arctan\left\{t\sqrt{\left(\frac{a-b}{a+b}\right)}\right\},\qquad \frac{1}{\sqrt{(b^2-a^2)}}\log\left\{\frac{\sqrt{(b+a)}+t\sqrt{(b-a)}}{\sqrt{(b+a)}-t\sqrt{(b-a)}}\right\},$$

where $t=\tan\frac{1}{2}x$, according as $a^2 \gtrless b^2$. If $a^2=b^2$ the integral reduces to a constant multiple of that of $\sec^2\frac{1}{2}x$ or $\operatorname{cosec}^2\frac{1}{2}x$, and its value may at once be written down. Deduce the forms of the integral when $a+b$ is negative.

4. Show that if y is defined in terms of x by means of the equation

$$(a+b\cos x)(a-b\cos y)=a^2-b^2,$$

where $a^2>b^2$, then as x varies from 0 to π one value of y varies from 0 to π. Show also that

$$\sin x=\frac{\sqrt{(a^2-b^2)}\sin y}{a-b\cos y},\qquad \frac{\sin x}{a+b\cos x}\frac{dx}{dy}=\frac{\sin y}{a-b\cos y};$$

and deduce that, if $0<x<\pi$,

$$\int\frac{dx}{a+b\cos x}=\frac{1}{\sqrt{(a^2-b^2)}}\arccos\left(\frac{a\cos x+b}{a+b\cos x}\right).$$

Show that this result agrees with that of Ex. 3.

5. Show how to integrate $1/(a+b\cos x+c\sin x)$. [Express $b\cos x+c\sin x$ in the form $\sqrt{(b^2+c^2)}\cos(x-\alpha)$.]

6. Integrate $(a+b\cos x+c\sin x)/(\alpha+\beta\cos x+\gamma\sin x)$.

[Determine λ, μ, ν so that

$$a+b\cos x+c\sin x\equiv\lambda+\mu(\alpha+\beta\cos x+\gamma\sin x)+\nu(-\beta\sin x+\gamma\cos x).$$

Then the integral is

$$\mu x+\nu\log(\alpha+\beta\cos x+\gamma\sin x)+\lambda\int\frac{dx}{\alpha+\beta\cos x+\gamma\sin x}.]$$

7. Integrate $1/(5+3\cos x)$, $1/(3-5\cos x)$, $1/(2-\sin x)$, $1/(1-\cos x+2\sin x)$, $(5+3\cos x-7\sin x)/(11-\cos x+\sin x)$.

8. Integrate $1/(a\cos^2 x+2b\cos x\sin x+c\sin^2 x)$. [The subject of integration may be expressed in the form $1/(A+B\cos 2x+C\sin 2x)$, where $A=\frac{1}{2}(a+c)$, $B=\frac{1}{2}(a-c)$, $C=b$: but the integral may be calculated more simply by putting $\tan x=t$, when we obtain

$$\int\frac{\sec^2 x\,dx}{a+2b\tan x+c\tan^2 x}=\int\frac{dt}{a+2bt+ct^2}.]$$

126. Integrals involving $\arcsin x$, $\arctan x$, **and** $\log x$. The integrals of the inverse sine and tangent and of the logarithm can easily be calculated by integration by parts. Thus

$$\int\arcsin x\,dx=x\arcsin x-\int\frac{x\,dx}{\sqrt{(1-x^2)}}=x\arcsin x+\sqrt{(1-x^2)},$$

$$\int\arctan x\,dx=x\arctan x-\int\frac{x\,dx}{1+x^2}=x\arctan x-\tfrac{1}{2}\log(1+x^2),$$

$$\int\log x\,dx=x\log x-\int dx=x(\log x-1).$$

It is easy to see that if we can find the integral of $y=f(x)$ we can always find that of $x=\phi(y)$, ϕ being the function inverse to f. For on making the substitution $y=f(x)$ we obtain

$$\int \phi(y)\,dy = \int x f'(x)\,dx = xf(x) - \int f(x)\,dx.$$

The reader should evaluate the integrals of $\arcsin y$ and $\arctan y$ in this way.

Any integrals of the form

$$\int P(x, \arcsin x)\,dx, \quad \int P(x, \log x)\,dx,$$

where P is a polynomial, can be found. Take the first form, for example. We have to calculate a number of integrals of the type $\int x^m (\arcsin x)^n\,dx$. Making the substitution $x=\sin y$ we obtain $\int y^n \sin^m y \cos y\,dy$, which can be found by the methods of § 124.

In the case of the second form we have to calculate a number of integrals of the form $\int x^m (\log x)^n\,dx$. Integrating by parts we obtain

$$\int x^m (\log x)^n\,dx = \frac{x^{m+1}(\log x)^n}{m+1} - \frac{n}{m+1}\int x^m (\log x)^{n-1}\,dx,$$

and it is evident that by repeating this process often enough we shall always arrive finally at the complete value of the integral*.

127. Areas of plane curves. One of the most important applications of the processes of integration which have been explained in the preceding sections is to the calculation of **areas** of plane curves. Suppose that P_0PP' (Fig. 53) is the graph of a continuous curve $y=\phi(x)$, P being the point (x, y) and P' the point $(x+h, y+k)$ and h being either positive or negative (positive in the figure).

The reader is of course familiar with the idea of an 'area,' and in particular with that of an area such as $ONPP_0$. This idea we

* A more general account of the problem of integration (§§ 111–126) will be found in Goursat's *Cours d'Analyse* or the author's tract quoted on p. 225. The reader may also be referred to the text-books of Profs. Lamb and Gibson, to Prof. Greenhill's *A Chapter in the Integral Calculus*, and a paper by Mr Bromwich in vol. xxxv of the *Messenger of Mathematics*.