by radicals. Irrational numbers that are not algebraic are called transcendental numbers. For a long time it was not known whether any transcendental numbers existed. Then finally, in 1844, Liouville exhibited an example of a transcendental number, similar to the one appearing in this exercise.

5.8 L'Hopital's Rule

In its simplest form this popular and useful rule is as follows. Let A be an open interval, and $f : A \to \mathbb{R}$, $g : A \to \mathbb{R}$ differentiable functions. Let c be a point in A, and suppose that $f(c) = g(c) = 0$, but that $g'(c) \neq 0$. Then we have

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{f'(c)}{g'(c)}.$$

To prove this we simply observe that for $x \neq c$ we have

$$\frac{f(x)}{g(x)} = \frac{\dfrac{f(x) - f(c)}{x - c}}{\dfrac{g(x) - g(c)}{x - c}}$$

and this tends to $f'(c)/g'(c)$ as x tends to c, by the definition of derivative and the rule for the limit of a quotient.

L'Hopital's rule can be framed in a more general form that vastly increases its usefulness. We no longer assume that the derivatives $f'(c)$ and $g'(c)$ exist. Instead we assume that $f'(x)/g'(x)$ tends to a limit as x tends to c. It is even more useful to take the limit as one-sided; after all, a two-sided limit is just a pair of one-sided limits that happen to be equal.

Proposition 5.13 (L'Hopital's rule for $0/0$) *Let* $f :]a, b[\to \mathbb{R}$, $g :]a, b[\to \mathbb{R}$ *be differentiable functions such that* $g'(x) \neq 0$ *for all* x *in the interval of definition. Suppose that*

$$\lim_{x\to a+} f(x) = 0, \quad \lim_{x\to a+} g(x) = 0, \quad \lim_{x\to a+} \frac{f'(x)}{g'(x)} = t.$$

Then

$$\lim_{x\to a+} \frac{f(x)}{g(x)} = t.$$

A similar conclusion holds for the limit $\lim_{x\to b-} f(x)/g(x)$.

The rule also holds if $a = -\infty$ *or* $b = \infty$; *or if* $t = \infty$ *or* $t = -\infty$.

Proof Consider first the right-hand limit at a in the case that a is not $-\infty$ and t is a finite number.

Let $\varepsilon > 0$. There exists $\delta > 0$, such that

$$\left|\frac{f'(x)}{g'(x)} - t\right| < \varepsilon$$

for all x that satisfy $a < x < a + \delta$. Let x and y satisfy $a < y < x < a + \delta$. By Cauchy's form of the mean value theorem there exists z between x and y, such that

$$\frac{f(x) - f(y)}{g(x) - g(y)} = \frac{f'(z)}{g'(z)}.$$

We deduce that for all such x and y we have

$$\left|\frac{f(x) - f(y)}{g(x) - g(y)} - t\right| < \varepsilon.$$

Let now $y \to a+$. We have that

$$\lim_{y \to a+} \frac{f(x) - f(y)}{g(x) - g(y)} = \frac{f(x)}{g(x)},$$

so that the inequality

$$\left|\frac{f(x)}{g(x)} - t\right| \leq \varepsilon$$

holds for all x that satisfy $a < x < a + \delta$. This proves the first assertion of L'Hopital's rule.

Next consider the case when $b = \infty$ and t is a finite number. We will determine $\lim_{x\to\infty} f(x)/g(x)$, the assumptions being that $\lim_{x\to\infty} f(x)$ and $\lim_{x\to\infty} g(x)$ are both 0, and $\lim_{x\to\infty} f'(x)/g'(x) = t$.

Let $\varepsilon > 0$. There exists K, such that

$$\left|\frac{f'(x)}{g'(x)} - t\right| < \varepsilon$$

for all $x > K$. Let $K < x < y$. There exists z between x and y, such that

$$\frac{f(x) - f(y)}{g(x) - g(y)} = \frac{f'(z)}{g'(z)},$$

and therefore, for all such x and y we have

$$\left|\frac{f(x) - f(y)}{g(x) - g(y)} - t\right| < \varepsilon.$$

Now let $y \to \infty$. We find that

$$\left| \frac{f(x)}{g(x)} - t \right| \leq \varepsilon$$

for all x that satisfy $x > K$, thus proving the rule in this case.

Consider the case when $t = \infty$, and a is a finite number. We shall determine $\lim_{x \to a+} f(x)/g(x)$, the assumptions being that $\lim_{x \to a+} f(x)$ and $\lim_{x \to a+} g(x)$ are both 0, and $\lim_{x \to a+} f'(x)/g'(x) = \infty$.

Let K be a real number and choose $\delta > 0$, such that

$$\frac{f'(x)}{g'(x)} > K$$

for all x that satisfy $a < x < a + \delta$. For all x and y that satisfy $a < y < x < a + \delta$ we obtain

$$\frac{f(x) - f(y)}{g(x) - g(y)} > K.$$

Let $y \to a+$. We deduce that $f(x)/g(x) \geq K$ for all x that satisfy $a < x < a + \delta$.

The reader should write out the proofs for all the remaining cases; each is similar to one of the cases treated above. The common feature is the use of Cauchy's mean value theorem. □

5.8.1 Using L'Hopital's Rule

There are two important things to bear in mind when one uses L'Hopital's rule. Firstly, $f(x)$ and $g(x)$ should both tend to 0 at the point where the limit of $f(x)/g(x)$ is sought. This is why we sometimes say that the rule resolves the indeterminate form $0/0$. Failure to observe this can lead to mistakes.

Secondly, we must observe the premise that $f'(x)/g'(x)$ has a limit. Thus it is not strictly correct, having first observed that $f(x)$ and $g(x)$ both tend to 0, to write that

$$\lim_{x \to a+} \frac{f(x)}{g(x)} = \lim_{x \to a+} \frac{f'(x)}{g'(x)},$$

before ascertaining that the limit on the right actually exists. For example there are cases when the limit on the left-hand side exists, but the limit on the right does not. Even so, we often write this, in the spirit of "let's wait and see," especially when the rule is used iteratively (more on this later) and it rarely leads to mistakes.

5.8.2 *Is There an Error in the Proof?*

The claim that

$$\lim_{y\to a+}\frac{f(x)-f(y)}{g(x)-g(y)}=\frac{f(x)}{g(x)}$$

appears in the proof of L'Hopital's rule. For this to be correct we must know that $g(x)\neq 0$. We are assuming that $\lim_{x\to a+} g(x)=0$ and there appears to be a danger that $g(x)$ might be 0 for some values of x near to a, maybe even for infinitely many values.

In fact we are safe on this score. One assumption was that $g'(x)\neq 0$ for $a<x<b$. Therefore given $\varepsilon>0$ we can (referring to the proof) find $\delta>0$ having the properties stated in the proof, but also such that $g(x)\neq 0$ for all x that satisfy $a<x<a+\delta$. This is because the equation $g(x)=0$ can have at most one solution in the open interval $]a,b[$, since otherwise, by Rolle's theorem, g' would have a zero in $]a,b[$.

The assumption that $g'(x)\ \not= 0$ for all x in its domain of definition is unnecessarily strong for applying L'Hopital's rule to calculate $\lim_{x\to a+} f(x)/g(x)$. Obviously it is enough that there should exist $h>0$, such that $g'(x)\neq 0$ for $a<x<a+h$.

5.8.3 *Geometric Interpretation of L'Hopital's Rule*

The two functions f and g define a parametric curve in the (x,y)-plane by letting $x=g(t)$, $y=f(t)$ for $a<t<b$. The assumptions that $\lim_{t\to a+} f(t)=\lim_{t\to a+} g(t)=0$ have the geometrical interpretation that the initial point of the curve is the coordinate origin $O=(0,0)$. Let $P(t)$ denote the point $\big(g(t),\,f(t)\big)$ in the plane, that is, the point on the curve with parameter t. Associated with the point $P(t)$ on the curve we can construct two lines. Firstly, the tangent at the point $P(t)$, corresponding to parameter t. Careful! The curve might cross itself. Secondly the chord joining $P(t)$ to the origin O.

L'Hopital's rule says the following: if the slope of the tangent has a limit as t tends to a (from the right), then the slope of the chord joining $P(t)$ to O has the same limit. The result is also valid if the limit is infinite; both chord and tangent then tend to a vertical position. The geometric interpretation of L'Hopital's rule is illustrated in Fig. 5.5.

5.8.4 *Iterative Use of L'Hopital's Rule: Taylor Polynomials Again*

If we wish to find the limit $\lim_{x\to a+} f(x)/g(x)$ using L'Hopital's rule we are directed to find the limit $\lim_{x\to a+} f'(x)/g'(x)$. This limit, too, may be found by L'Hopital's rule if it happens that $\lim_{x\to a+} f'(x)=\lim_{x\to a+} g'(x)=0$; then we are directed

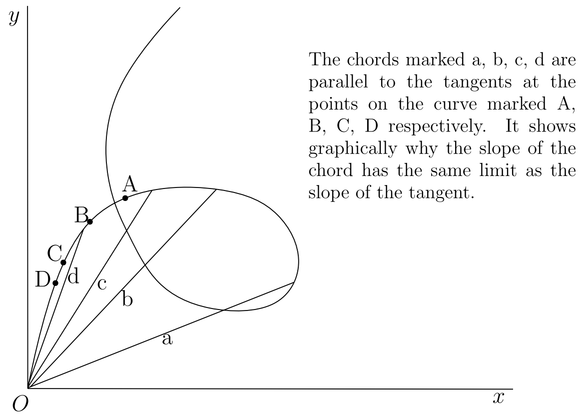

Fig. 5.5 L'Hopital's rule at a glance

to the limit $\lim_{x\to a+} f''(x)/g''(x)$. Again it could happen that $\lim_{x\to a+} f''(x) = \lim_{x\to a+} g''(x) = 0$. As long as numerator and denominator have the limit 0 we may differentiate them, until a limit is found that we can easily compute.

The iterative use of L'Hopital's rule gives an easy proof of the approximation property of Taylor polynomials stated in Sect. 5.7 under the heading "Higher derivatives and Taylor polynomials".

Proposition 5.14 *Let $f : A \to \mathbb{R}$, where A is an open interval, and let $c \in A$. Assume that the derivatives $f'(c)$, ..., $f^{(m)}(c)$ all exist and define*

$$E(h) = f(c+h) - \left(f(c) + \frac{1}{1!}f'(c)h + \frac{1}{2!}f''(c)h^2 + \cdots + \frac{1}{m!}f^{(m)}(c)h^m \right)$$

for all h such that $|h|$ is sufficiently small. Then

$$\lim_{h\to 0} \frac{E(h)}{h^m} = 0.$$

Note that h can be positive or negative, but we require that $c+h$ is in the interval A. That is why we want $|h|$ to be "sufficiently small".

The assumption that f has derivatives at c up to order m means that f is $(m-1)$-times differentiable in some open interval containing c, and $f^{(m-1)}$ is differentiable at c.

Proof of the Proposition Differentiating $E(h)$ repeatedly with respect to h we obtain, for $j = 1, ..., m-1$ and for all h such that $|h|$ is sufficiently small,

$$E^{(j)}(h) = f^{(j)}(c+h) - \left(f^{(j)}(c) + \frac{1}{1!}f^{(j+1)}(c)h + \cdots + \frac{1}{(m-j)!}f^{(m)}(c)h^{m-j} \right),$$

from which we see that $E^{(j)}(0) = 0$ for $j = 1, ..., m-1$. We also have (convenient to use Leibniz's notation here)

$$\frac{d^j}{dh^j}h^m = \frac{m!}{(m-j)!}h^{m-j} \text{ for } j = 0, 1, 2, ..., m,$$

so that

$$\left.\frac{d^j}{dh^j}h^m\right|_{h=0} = 0 \text{ for } j = 0, 1, 2, ..., m-1.$$

Using L'Hopital's rule iteratively (with a "wait and see" approach to the existence of the limits) now gives

$$\lim_{h\to 0}\frac{E(h)}{h^m} = \lim_{h\to 0}\frac{E^{(m-1)}(h)}{m!h} = \frac{1}{m!}\lim_{h\to 0}\frac{f^{(m-1)}(c+h) - f^{(m-1)}(c) - f^{(m)}(c)h}{h},$$

and this is 0 by definition of derivative (and it also tells us that $E^{(m)}(0) = 0$). This completes the proof. □

Let us write $x - c$ for h in the formula for $E(h)$. We obtain the conclusion

$$\lim_{x\to c}\frac{f(x) - P_m(x, c)}{(x-c)^m} = 0.$$

This describes admirably how the approximation to $f(x)$ by the Taylor polynomial $P_m(x, c)$ improves sharply as x approaches c. On the other hand there is no reason to think that the approximation improves if we hold x fixed and increase m (assuming we have the derivatives). This question is partly settled by Taylor's theorem proper in a later chapter.

5.8.5 Application to Maxima and Minima

If the derivative of f at c is zero, the examination of higher derivatives at c can sometimes resolve the question as to whether c is a local maximum point or a local minimum point.

Proposition 5.15 *Let* $f :]a, b[\to \mathbb{R}$ *and let* $a < c < b$. *Assume that* f *is* $(m-1)$-*times differentiable, that* $f^{(j)}(c) = 0$ *for* $j = 1, 2, ...m-1$, *but that* $f^{(m)}(c)$ *exists and is* not 0. *In addition to all this assume that* m *is an* even number. *The following conclusions then hold:*

(1) If $f^{(m)}(c) > 0$ *then* c *is a strict local minimum point.*
(2) If $f^{(m)}(c) < 0$ *then* c *is a strict local maximum point.*

Proof By Proposition 5.14 we have

$$f(c+h) - f(c) = \frac{1}{m!}f^{(m)}(c)h^m + E(h)$$

where the error term $E(h)$ satisfies

$$\lim_{h\to 0} \frac{E(h)}{h^m} = 0.$$

For $h \neq 0$ we can write

$$\frac{f(c+h) - f(c)}{h^m} = \frac{1}{m!} f^{(m)}(c) + \frac{E(h)}{h^m}$$

and we know that $f^{(m)}(c) \neq 0$. Since m is even we must have $h^m > 0$, both for $h > 0$ and for $h < 0$. We conclude that there exists $\delta > 0$, such that $f(c+h) - f(c) \neq 0$ and has the same sign as $f^{(m)}(c)$, for all h that satisfy $0 < |h| < \delta$. This is precisely the sought-for conclusion. □

5.8.6 More on L'Hopital's Rule: The ∞/∞ Version

Sometimes we consider L'Hopital's rule as resolving the indeterminate form $0/0$, an expression that is really quite meaningless. Since we are indulging in meaninglessness we might suggest some other indeterminate forms, for example

$$\frac{\infty}{\infty}, \quad 0.\infty, \quad 0^0, \quad \infty - \infty.$$

These can often be resolved by some judicious manipulations combined with L'Hopital's rule. However there is a version of the rule directly applicable to ∞/∞ and, as we shall see, it turns out to be very useful.

Proposition 5.16 *Let* $f :]a, b[\to \mathbb{R}$, $g :]a, b[\to \mathbb{R}$ *be differentiable functions such that* $g'(x) \neq 0$ *for all* x *in its domain of definition. Assume that*

$$\lim_{x\to a+} g(x) = \infty, \quad \lim_{x\to a+} \frac{f'(x)}{g'(x)} = t.$$

Then

$$\lim_{x\to a+} \frac{f(x)}{g(x)} = t.$$

A similar conclusion holds for $\lim_{x\to b-} f(x)/g(x)$.

The rule also holds if $a = -\infty$ *or* $b = \infty$*; or if* $t = \infty$ *or* $t = -\infty$.

Note that we made no assumption about $\lim_{x\to a+} f(x)$. This is not a mistake.

Proof of the Proposition We shall only consider the case when t and a are finite numbers. The other cases are left to the reader to complete.

Let $\varepsilon > 0$. Since $\lim_{x\to a+} f'(x)/g'(x) = t$, there exists $\delta_1 > 0$, such that

$$\left|\frac{f'(z)}{g'(z)} - t\right| < \varepsilon$$

for all z that satisfy $a < z < a + \delta_1$.

Let $a < x < y < a + \delta_1$. It follows by Cauchy's form of the mean value theorem, that

$$\left|\frac{f(x) - f(y)}{g(x) - g(y)} - t\right| < \varepsilon,$$

which we rewrite in the form

$$t - \varepsilon < \frac{f(x) - f(y)}{g(x) - g(y)} < t + \varepsilon.$$

Now keep y fixed (if desired we could fix y as $a + \frac{1}{2}\delta_1$, but the thing we do require is that $a < x < y < a + \delta_1$). Since $\lim_{x\to a+} g(x) = \infty$ we find that $g(x) > 0$ and $g(x) - g(y) > 0$ when x is sufficiently close to a, for example for $a < x < a + \delta_2$, and then we have, for $a < x < y < \delta_1$ and $a < x < a + \delta_2$, that

$$\big(g(x) - g(y)\big)(t - \varepsilon) < f(x) - f(y) < \big(g(x) - g(y)\big)(t + \varepsilon).$$

Dividing by $g(x)$ (which is positive) gives

$$\left(1 - \frac{g(y)}{g(x)}\right)(t - \varepsilon) < \frac{f(x) - f(y)}{g(x)} < \left(1 - \frac{g(y)}{g(x)}\right)(t + \varepsilon),$$

and therefore

$$\frac{f(y)}{g(x)} + \left(1 - \frac{g(y)}{g(x)}\right)(t - \varepsilon) < \frac{f(x)}{g(x)} < \frac{f(y)}{g(x)} + \left(1 - \frac{g(y)}{g(x)}\right)(t + \varepsilon).$$

As $x \to a+$ the left-hand member of the inequalities tends to $t - \varepsilon$ and the right-hand member to $t + \varepsilon$ (recall that we keep y constant). Hence there exists $\delta_3 > 0$, such that the left-hand member is above $t - 2\varepsilon$ and the right-hand member below $t + 2\varepsilon$ for all x that satisfy $a < x < a + \delta_3$. Let $\delta = \min(\delta_2, \delta_3)$. If $a < x < a + \delta$ we find

$$t - 2\varepsilon < \frac{f(x)}{g(x)} < t + 2\varepsilon.$$

This says that $\lim_{x\to a+} f(x)/g(x) = t$ and concludes the proof of the first claim. The proofs of the remaining claims are left to the reader. □

The proof raises some interesting speculation about the meaning of "for each ε there exists δ". It is too simple to say that δ is supposed to be a function of ε. In the above proof we first chose δ_1, in a non-explicit fashion, from the set of all possible numbers that would work for the limit $\lim_{x\to a+} f'(x)/g'(x)$. Then y was chosen rather arbitrarily. A workable value was defined in an aside but that was not really necessary. Finally we found a δ that worked.

Where desired and possible we can try to define the quantities we use by functions (such as using the max function). But at times we have to say, as in effect we did at the beginning of the above proof, "here is a set (of usable δ's), known to be non-empty; let us choose one". Sometimes it is simply not very helpful to try to see δ as an explicitly computable function of ε.

5.8.7 Exercises

1. Calculate the following limits:

   (a) $\displaystyle\lim_{x\to 1} \frac{x^4 - x^3 - x + 1}{x^4 - 3x^3 + 2x^2 + x - 1}$

   (b) $\displaystyle\lim_{x\to 0} \frac{\sqrt{1+x} - 1}{\sqrt[3]{1+x} - 1}$

   (c) $\displaystyle\lim_{x\to \infty} \sqrt{x^2 + x} - x.$

2. Calculate the following limits. Use your school knowledge of the circular functions $\sin x$ and $\cos x$ and their derivatives (or refer to Sect. 5.1, Exercise 2).

   (a) $\displaystyle\lim_{x\to 0} \frac{1}{x} - \frac{1}{\sin x}$

   (b) $\displaystyle\lim_{x\to 0} \frac{1}{x^2} - \frac{1}{x \sin x}$

   (c) $\displaystyle\lim_{x\to 0} \frac{1}{6x} + \frac{1}{x^3} - \frac{1}{x^2 \sin x}.$

3. Exploiting only two properties of the exponential function:

   $$\lim_{x\to\infty} e^x = \infty \quad \text{and} \quad \frac{d}{dx} e^x = e^x,$$

   show that for any natural number n we have $\lim_{x\to\infty} e^x/x^n = \infty$.

   Note. The conclusion demonstrates the proverbial growth of the exponential function in a graphic way; it overpowers any polynomial.

4. Let f be twice differentiable in an interval A, let a be a point in A and suppose that $f''(a) \neq 0$. Show that the tangent to the graph $y = f(x)$ at the point $(a, f(a))$ does not cross the graph at $(a, f(a))$. Show, in addition, that there exists $\delta > 0$, such that the tangent and the graph have no common point in the interval $]a - \delta, a + \delta[$, except at $x = a$.

5. Suppose the function f is differentiable in an interval A and let $a \in A$. Let λ and μ be distinct numbers. Show that

$$f'(a) = \lim_{h \to 0} \frac{f(a + \lambda h) - f(a + \mu h)}{(\lambda - \mu)h}.$$

Note. A case important for *numerical differentiation* is $\lambda = 1, \mu = -1$. More general formulas are known, approximating the first, and higher, derivatives. See the next exercises.

6. Suppose the function f is differentiable in an interval A and let $a \in A$. Show that

$$f'(a) = \lim_{h \to 0} \frac{-f(a + 2h) + 8f(a + h) - 8f(a - h) + f(a - 2h)}{12h}.$$

7. Suppose the function f is twice differentiable in an interval A and let $a \in A$. Show that

$$f''(a) = \lim_{h \to 0} \frac{f(a + h) - 2f(a) + f(a - h)}{h^2}.$$

8. (a) Show that for all positive integers n

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} k^j = \begin{cases} 0, & j = 0, 1, ..., n - 1 \\ (-1)^n n! & j = n. \end{cases}$$

   Hint. Expand $(1 - x)^n$ by the binomial rule. Repeatedly differentiate, but with a twist.

   (b) Suppose the function f is n times differentiable in an interval A and let $a \in A$. Show that

$$f^{(n)}(a) = \lim_{h \to 0} \frac{1}{h^n} \sum_{k=0}^{n} (-1)^{k+n} \binom{n}{k} f(a + kh).$$

9. We can define the left derivative of f at c, denoted by $D_l f(c)$, and the right derivative $D_r f(c)$, in the obvious way:

$$D_l f(c) = \lim_{x \to c-} \frac{f(x) - f(c)}{x - c}, \quad D_r f(c) = \lim_{x \to c+} \frac{f(x) - f(c)}{x - c}$$

   when these limits exist. Now suppose that f is differentiable in an interval $]c - \alpha, c[$, continuous in $]c - \alpha, c]$ and the limit $\lim_{x \to c-} f'(x)$ exists and is a finite number A. Show that $D_l f(c)$ exists and equals A. A similar result holds for the right derivative.

   Show that the result also holds if $A = \infty$ or $-\infty$, if we allow a derivative to be infinite (the definition should be obvious).

10. The previous exercise has an interesting consequence. Suppose that f is differentiable everywhere in an open interval A. Show that discontinuities of f', if there are any, are never jump discontinuities.

    Note. f' can be discontinuous. An example was exhibited in Sect. 5.5, Exercise 4.