2.3. Bounded sequences

A sequence $a_1, a_2, \ldots$ is called *bounded* if the set of its terms is bounded, i. e. if there exists a number M such that the inequality $|a_n| < M$ holds for all values of n. In other words, the inequality $-M < a_n < M$ is satisfied for all terms of the sequence.

The geometrical interpretation of this condition is that the whole sequence (treated as a set on the plane) is contained between two straight lines $y = M$ and $y = -M$.

We say also that a sequence is *bounded above* if a number M exists such that $a_n < M$ for all n, i.e. that the sequence lies below the straight line $y = M$. The notion of a sequence *bounded below* can be defined analogously. Clearly, a sequence which is bounded above and below is simply a bounded sequence.

EXAMPLES. The sequence $0, 1, 0, 1, \ldots$ is bounded. The sequence of positive integers is unbounded, though it is bounded below. The sequence $1, -1, 2, -2, 3, -3, \ldots$ is neither bounded above nor bounded below.

THEOREM. *Every convergent sequence is bounded.*

Indeed, let us assume that the equation (2) holds and let us substitute the value 1 for ε. Hence a number k exists such that we have $|a_n - g| < 1$ for $n > k$. Since $|a_n| - |g| \leqslant |a_n - g|$ (cf. § 1 (13)), we have $|a_n| < |g| + 1$. Let us denote by M a number greater than any among the following $k+1$ numbers: $|a_1|, |a_2|, \ldots, |a_k|, |g|+1$. Since the last one is greater than $|a_{k+1}|, |a_{k+2}|$ etc., we get $M > |a_n|$ for each n. Thus, the sequence is bounded.

2.4. Operations on sequences

THEOREM. *Assuming the sequences $a_1, a_2, \ldots$ and $b_1, b_2, \ldots$ to be convergent, the following four formulae hold* (¹):

(6) $\lim(a_n + b_n) = \lim a_n + \lim b_n$,

(7) $\lim(a_n - b_n) = \lim a_n - \lim b_n$,

(¹) To simplify the symbolism we shall often omit the equality $n = \infty$ under the sign of lim.

(8) $\lim(a_n \cdot b_n) = \lim a_n \cdot \lim b_n$,

(9) $\lim \frac{a_n}{b_n} = \frac{\lim a_n}{\lim b_n}$ (*when* $\lim b_n \neq 0$).

This means that under our assumptions the limit of the sum exists and is equal to the sum of limits, the limit of the difference exists and is equal to the difference of limits etc.

Proof. Let us write $\lim a_n = g$ and $\lim b_n = h$. A number $\varepsilon > 0$ let be given. Hence a number k exists such that the inequalities $|a_n - g| < \varepsilon/2$ and $|b_n - h| < \varepsilon/2$ hold for $n > k$. We add both these inequalities under the sign of absolute value. We obtain

$$|(a_n + b_n) - (g + h)| < \varepsilon .$$

This means that the sequence with general term $c_n = a_n + b_n$ is convergent to the limit $g + h$. Thus, we have proved the formula (6).

In particular, if b_n takes a constant value: $b_n = c$, formulae (6) and (3) imply:

(10) $$\lim(a_n + c) = c + \lim a_n .$$

Now, we shall prove the formula (8). We have to "estimate" the difference $|a_n b_n - gh|$. To be able to apply the convergence of the sequences $a_1, a_2, \ldots$ and $b_1, b_2, \ldots$ we transform this difference as follows:

$$a_n b_n - gh = a_n b_n - a_n h + a_n h - gh = a_n(b_n - h) + h(a_n - g) .$$

Since the sequence $a_1, a_2, \ldots$ is convergent, it is bounded and so a number M exists such that $|a_n| < M$. Applying to the last equation the formulae for the absolute value of a sum and of a product we get:

$$\begin{aligned} |a_n b_n - gh| &\leqslant |a_n(b_n - h)| + |h(a_n - g)| \\ &\leqslant M \cdot |b_n - h| + |h| \cdot |a_n - g| . \end{aligned}$$

Now, let us take a number $\eta > 0$ independently of ε. Hence a number k exists such that we have $|a_n - g| < \eta$

and $|b_n - h| < \eta$ for $n > k$. Thus,

$$|a_n b_n - gh| < M\eta + |h|\eta = (M + |h|)\eta .$$

Till now we have not assumed anything about the positive number η. Let us now assume that $\eta = \varepsilon/(M+|h|)$. So we conclude that the inequality $|a_n b_n - gh| < \varepsilon$ holds for $n > k$. Thus, we have proved the formula (8).

In particular, if we write $b_n = c$ we get

$$\lim(c \cdot a_n) = c \lim a_n , \tag{11}$$

$$\lim(-a_n) = -\lim a_n , \tag{12}$$

where the formula (12) follows from (11) by the substitution $c = -1$.

Formulae (6) and (12) imply the formula (7), for

$$\lim(a_n - b_n) = \lim[a_n + (-b_n)]$$
$$= \lim a_n + \lim(-b_n) = \lim a_n - \lim b_n .$$

Before proceeding to the proof of the formula (9), we shall prove the following special case of this formula:

$$\lim \frac{1}{b_n} = \frac{1}{\lim b_n} \quad (\textit{when } \lim b_n \neq 0) . \tag{13}$$

First, we note that for sufficiently large n the inequality $b_n \neq 0$ holds. We shall prove an even stronger statement: we have $|b_n| > \frac{1}{2}|h|$ for sufficiently large n. Indeed, since $\frac{1}{2}|h| > 0$, a number k exists such that $|b_n - h| < \frac{1}{2}|h|$ for $n > k$. Hence,

$$|h| - |b_n| \leqslant |h - b_n| < \tfrac{1}{2}|h| \quad \text{and thus} \quad |b_n| > \tfrac{1}{2}|h| .$$

To prove the formula (13), the difference

$$\left|\frac{1}{b_n} - \frac{1}{h}\right| = \left|\frac{h - b_n}{h \cdot b_n}\right| = \frac{|h - b_n|}{|h| \cdot |b_n|}$$

has to be estimated.

But for sufficiently large n we have $|h - b_n| < \eta$ and $|b_n| > \frac{1}{2}|h|$, i. e. $1/|b_n| < 2/|h|$. Thus,

$$\left|\frac{1}{b_n} - \frac{1}{h}\right| < \frac{2\eta}{h^2} .$$

Writing $\eta = \frac{1}{2}\varepsilon h^2$, we get

$$\left|\frac{1}{b_n} - \frac{1}{h}\right| < \varepsilon ,$$

whence the formula (13) follows.

The formula (9) follows from (8) and (13):

$$\lim \frac{a_n}{b_n} = \lim a_n \cdot \frac{1}{b_n} = \lim a_n \cdot \lim \frac{1}{b_n} = \frac{\lim a_n}{\lim b_n} .$$

Remarks. (α) We have assumed that the sequences $\{a_n\}$ and $\{b_n\}$ are convergent. This assumption is essential, for it may happen that the sequence $\{a_n + b_n\}$ is convergent, although both the sequences $\{a_n\}$ and $\{b_n\}$ are divergent; then the formula (6) cannot be applied. As an example one can take: $a_n = n$, $b_n = -n$.

However, if the sequence $\{a_n + b_n\}$ and one of the two sequences, e. g. the sequence $\{a_n\}$ are convergent, then the second one is also convergent. For $b_n = (a_n + b_n) - a_n$, and so the sequence $\{b_n\}$ is convergent as a difference of two convergent sequences.

Analogous remarks may be applied to the formulae (7)–(9).

(β) In the definition of a sequence we have assumed that the enumeration of the elements begins with 1. It is convenient to generalize this definition assuming that the enumeration begins with an arbitrary positive integer (and even with an arbitrary integer), e.g. with 2, 3 or another positive integer. So is e.g. in the proof of the formula (13). We have proved that $b_n \neq 0$ beginning with a certain k. Thus, the sequence $\frac{1}{b_n}$ is defined just beginning with this k (for if $b_n = 0$, then $\frac{1}{b_n}$ does not mean any number).

This remark is connected with the following property of sequences, easy to prove: *the change of a finite number of terms of a sequence has influence neither on the con-*

vergence of the sequence nor on its limit. The same can be said of the addition or omission of a finite number of terms in a sequence.

EXAMPLES. Find $\lim\limits_{n=\infty}\dfrac{6n+2}{7n-3}$. In this case we cannot apply the formula (9) directly, since neither the numerator nor the denominator are convergent, as n tends to ∞. However, we can transform the general term of the sequence $a_n = \dfrac{6n+2}{7n-3}$ to become a quotient of two expressions each of which has a limit. For this purpose it is sufficient to divide the numerator and the denominator by n. Then we obtain $a_n = \dfrac{6+2/n}{7-3/n}$ and we may apply the formula (9). Since

$$\lim_{n=\infty}\left(6+\frac{2}{n}\right)=6 \quad \text{and} \quad \lim_{n=\infty}\left(7-\frac{3}{n}\right)=7,$$

we have

$$\lim_{n=\infty} a_n = \frac{6}{7}.$$

Similarly,

$$\lim_{n=\infty}\frac{n+1}{n^2+1}=\lim_{n=\infty}\frac{\frac{1}{n}+\frac{1}{n^2}}{1+\frac{1}{n^2}}=0,$$

for

$$\lim_{n=\infty}\frac{1}{n}=0=\lim_{n=\infty}\frac{1}{n^2}.$$

2.5. Further properties of the limit

Suppose that a sequence $\{a_n\}$ is convergent. Then the sequence $\{|a_n|\}$ is convergent, too, and

$$(14) \qquad \lim|a_n| = |\lim a_n|.$$

Let $\lim a_n = g$. Then we have $|a_n - g| < \varepsilon$ for sufficiently large n. Thus,

$$|a_n|-|g| \leqslant |a_n-g| < \varepsilon \quad \text{and} \quad |g|-|a_n| \leqslant |g-a_n| < \varepsilon,$$

whence (cf. § 1 (17)): $||a_n|-|g|| < \varepsilon$. Thus, the formula (14) is proved.

Assuming the sequences $\{a_n\}$ and $\{b_n\}$ to be convergent, the following relation holds:

(15) *the inequality* $a_n \leqslant b_n$ *implies* $\lim a_n \leqslant \lim b_n$.

In particular, if the sequence $\{c_n\}$ is convergent, then

(16) *the condition* $c_n \geqslant 0$ *implies* $\lim c_n \geqslant 0$.

We shall prove first the last formula. Let $\lim c_n = h$, and let us further assume that $h < 0$, i.e. that $-h > 0$. Then we have $|c_n - h| < -h$ for sufficiently large n, and hence $c_n - h < -h$, whence $c_n < 0$, which contradicts our assumption.

Applying the formula (16) we shall prove now the formula (15).

We put $b_n - a_n = c_n$. Since $a_n \leqslant b_n$, we have $c_n \geqslant 0$ and thus, in the limit, $\lim c_n \geqslant 0$. Moreover, (7) implies:

$$\lim c_n = \lim b_n - \lim a_n,$$

hence

$$\lim b_n - \lim a_n \geqslant 0, \quad \text{i.e.} \quad \lim a_n \leqslant \lim b_n.$$

Remark. In the formulation of the relation (16), the inequality $\geqslant$ cannot be replaced by $>$ (similarly, in (15) one cannot replace $\leqslant$ by $<$). For example the sequence $c_n = 1/n$ satisfies the inequality $c_n > 0$, but $\lim c_n = 0$.

Thus we see that the relations $\leqslant$ and $\geqslant$ "remain true in the limit", but the relations $<$ and $>$ do not possess this property.

We next prove the *formula of the double inequality*:

(17) *If* $a_n \leqslant c_n \leqslant b_n$ *and if* $\lim a_n = \lim b_n$, *then the sequence* $\{c_n\}$ *is convergent and* $\lim c_n = \lim a_n = \lim b_n$.

Suppose that $\lim a_n = g = \lim b_n$ and let $\varepsilon > 0$. Then we have $|a_n - g| < \varepsilon$ and $|b_n - g| < \varepsilon$ for sufficiently large n. According to the assumption,

$$a_n - g \leqslant c_n - g \leqslant b_n - g,$$

$$\text{and} \quad -\varepsilon < a_n - g \text{ and } b_n - g < \varepsilon;$$

hence $-\varepsilon < c_n - g < \varepsilon$, i. e. $|c_n - g| < \varepsilon$, whence $\lim c_n = g$.

(18) *If* $\lim|a_n| = 0$, *then the sequence* $\{a_n\}$ *is convergent and* $\lim a_n = 0$.

For, we have then $-|a_n| \leqslant a_n \leqslant |a_n|$ and

$$\lim(-|a_n|) = 0 = \lim|a_n|.$$

2.6. Subsequences

Let a sequence $a_1, a_2, \ldots, a_n, \ldots$ and an increasing sequence of positive integers $m_1, m_2, \ldots, m_n, \ldots$ be given. The sequence

$$b_1 = a_{m_1}, \quad b_2 = a_{m_2}, \quad \ldots, \quad b_n = a_{m_n}, \quad \ldots$$

is called a *subsequence of the sequence* $a_1, a_2, \ldots, a_n, \ldots$

For example the sequence $a_2, a_4, \ldots, a_{2n}, \ldots$ is a subsequence of the sequence $a_1, a_2, \ldots$ Yet, the sequence $a_1, a_1, a_2, a_2, \ldots$ is not a subsequence of this sequence, since in this case the indices do not form an increasing sequence.

We have the general formula

(19) $$m_n \geqslant n.$$

This is obvious for $n = 1$, i.e. we have $m_1 \geqslant 1$ (since m_1 is a positive integer). Applying the principle of induction, let us assume that the formula (19) holds for a given n. Then we have $m_{n+1} > m_n \geqslant n$, whence $m_{n+1} \geqslant n+1$. So we have obtained the formula (19) for $n+1$. Thus, the formula (19) is true for each n.

According to our definition, every sequence is its own subsequence. We can say in general that every subsequence is obtained from the sequence by omitting a number of elements in this sequence (this number may be finite, infinite or zero). Hence it follows also that a subsequence $\{a_{m_{k_n}}\}$ of a subsequence $\{a_{m_n}\}$ is a subsequence of the sequence $\{a_n\}$.

THEOREM 1. *A subsequence of a convergent sequence is convergent to the same limit. In other words,*

(20) *if* $\lim\limits_{n=\infty} a_n = g$ *and if* $m_1 < m_2 < \ldots$, *then* $\lim\limits_{n=\infty} a_{m_n} = g$.