So how would you write the proof? Certainly the proof would begin with selecting a generic sequence and making a statement about the properties the sequence is assumed to have, that is, its being monotone increasing and bounded above. Then, the proof would proceed to justify the existence of the least upper bound for the set of terms of the sequence; that will give you the target value of L. Then, as with most proofs about limits, it would select a value for $\epsilon > 0$. Unlike the limit proofs earlier in this chapter, one cannot immediately state a value for N. The existence of N must be proved as discussed in the previous paragraph. Finally, the properties of the sequence can be brought together to show $|a_n - L| < \epsilon$ for all $n > N$. Here is one possible proof.

PROOF: A monotone increasing sequence that is bound above converges.

- Let $<a_j>$ be a monotone increasing sequence of real numbers that is bounded above.
- Since the set of terms $A = \{a_j \,|\, j \in \mathbb{N}\}$ contains a_1, it is nonempty, and since it is bounded above, the Completeness Axiom guarantees that A has a least upper bound, L.
- Given $\epsilon > 0$, the number $L - \epsilon$ is less than L. Since L is the least upper bound of A, $L - \epsilon$ is not an upper bound of A. Thus, there is an $N \in \mathbb{N}$ such that the term a_N is in A and is larger than $L - \epsilon$.
- Select an $n > N$.
- Because $<a_j>$ is monotone increasing, $a_n \geq a_N$. Because L is an upper bound for A, $a_n \leq L$. Therefore, $L - \epsilon < a_N \leq a_n \leq L$, and $|a_n - L| < |(L - \epsilon) - L| = \epsilon$.
- This proves that the sequence $<a_j>$ has limit L and that $<a_j>$ converges.

Note that the proof needs to refer to the sequence $<a_n>$ as well as a particular element of the sequence a_n. It could be confusing to the proof reader to use the variable n in both contexts here, especially since the sequence notation $<a_n>$ is used after the choice of a specific value of n is made. That is the reason the proof changed to using the variable j to refer to a generic term index. Then, it could refer to a specific term using index n without confusing the two uses.

There is also a theorem stating that a monotone decreasing sequence that is bounded below converges. The proof of this is left as an exercise.

As an illustration of the usefulness of the above result, consider a sequence defined recursively by $a_1 = 2$, and for $n \geq 1$, $a_{n+1} = \sqrt{a_n + 12}$. That is, $a_1 = 2$, $a_2 = \sqrt{a_1 + 12} = \sqrt{14}$, $a_3 = \sqrt{\sqrt{14} + 12}$, and so forth. One can prove that this sequence converges by showing that the sequence is both monotone increasing and bounded above. Indeed, both of these facts can be established by **mathematical induction**. The reader is likely already familiar with proofs by mathematical induction, but this is an appropriate opportunity to review the method and its merits.