3.4 Limits at Infinity

The definitions given in the last two sections do not make sense when the real number that x approaches, a, is replaced by infinity. Infinity, of course, is not an element of the real numbers, $\mathbb{R}$, but it does make sense to ask whether a function approaches a limit when x increases without bound, that is, as x approaches infinity. When one writes $\lim_{x\to\infty} f(x) = L$, one is thinking that $f(x)$ is getting close to the real number L as x increases without bound. But it does not make sense to measure how close x is to infinity by choosing a $\delta > 0$ so that when x is within δ of infinity, $f(x)$ is close to L. Since infinity is not a real number, one cannot measure the distance from the real number, x, to infinity, even less expect x to get within δ of infinity. So how does one quantify "getting closer to infinity?" The answer lies in the phrase "increases without bound" which suggests that for any bound, N, you could place on the size of x, the value of x can be made to be greater than that bound. Thus, instead of selecting a $\delta > 0$ and requiring $0 < |x - a| < \delta$, one chooses a number $N \in \mathbb{R}$ and requires $x > N$. This allows the following definition. Suppose that the function f is defined for all $x > K$ for some real number K. Then the **limit of f as x approaches infinity is L, $\lim_{x\to\infty} f(x) = L$**, means that for every $\epsilon > 0$ there exists an $N \in \mathbb{R}$ such that for every $x > N$, it follows that $|f(x) - L| < \epsilon$ (Fig. 3.5). Now consider how one might write a proof of a limit at infinity. For example, consider proving the limit $\lim_{x\to\infty} \frac{x}{x^2+6} = 0$. Here $f(x) = \frac{x}{x^2+6}$ and $L = 0$. As with other limit proofs, the goal is to arrange that $|f(x) - L| < \epsilon$ for an arbitrarily chosen $\epsilon > 0$. Again, you can work backwards. Since $|f(x) - L| = \left|\frac{x}{x^2+6}\right|$, as long as $x > 0$, it would follow that $\left|\frac{x}{x^2+6}\right| < \frac{x}{x^2} = \frac{1}{x}$. Thus, there is an expression, $\frac{1}{x}$, which is larger than $|f(x) - L|$ for all suitably large values of x. This will help because if you can assure that $\frac{1}{x}$ is less than ϵ, it will follow that $|f(x) - L|$ is also less than ϵ. It would not have been helpful to exhibit an expression that was always less than $|f(x) - L|$ because making that expression small would not imply that $|f(x) - L|$ is small. Now, if $x > \frac{1}{\epsilon}$, it follows that $\frac{1}{x} < \epsilon$ suggesting that $\frac{1}{\epsilon}$ is a suitable value for N.

PROOF: $\lim_{x\to\infty} \frac{x}{x^2+6} = 0$

- Let $f(x) = \frac{x}{x^2+6}$.
- Given $\epsilon > 0$, let $N = \frac{1}{\epsilon}$.
- Select x such that $x > N > 0$.
- Then $x > \frac{1}{\epsilon}$ implies $\epsilon > \frac{1}{x} = \frac{x}{x^2} > \frac{x}{x^2+6} = \left|\frac{x}{x^2+6} - 0\right| = |f(x) - 0|$.
- Therefore, $\lim_{x\to\infty} \frac{x}{x^2+6} = 0$.

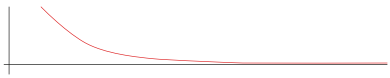

Fig. 3.5 Approaching a limit as $x \to \infty$