are all upper bounds of S, but 5 is the least upper bound of S. Also, -2, 0, and $\frac{1}{2}$ are all lower bounds of S, but 1 is the greatest lower bound of S. One often uses the notation l.u.b.(S) or $\sup(S)$ to represent the least upper bound or **supremum** of S and g.l.b.(S) or $\inf(S)$ to represent the greatest lower bound or **infimum** of S.

Axioms for the Real Numbers

The real numbers, $\mathbb{R}$, is an ordered field that satisfies **The Completeness Axiom**:

Every nonempty set $S \subseteq \mathbb{R}$ which is bounded above has a least upper bound in $\mathbb{R}$.

Note, for example, that the set $S = \{x \in \mathbb{Q} \mid x^2 < 7\}$ is a nonempty subset of $\mathbb{Q}$ which is bounded above by 4, 3, and 2.7, but there is no element of $\mathbb{Q}$ which is a least upper bound of S. The set of real numbers, though, does contain a least upper bound of S, namely $\sqrt{7}$. The Completeness Axiom is sometimes called the **Least Upper Bound Principle**. The Completeness Axiom comes up frequently in proofs about the real numbers to show that numbers with particular properties exist. For example, consider the two theorems, the **Archimedean Principle** and the **Existence of Square Roots**. Both of these theorems are easily understood, but they cannot be proved without using the Completeness Axiom.

The Archimedean Principle states that for every real number r there is a natural number greater than r. It can be proved using a *proof by contradiction*. The proof makes the assumption that there is a real number greater than every natural number and uses this to derive a contradiction, a statement that is false. Because one cannot derive a false statement from a true statement, the assumption most recently made in the proof must be a false statement, and you can conclude that no real number exists that is greater than every natural number.

PROOF (Archimedean Principle): If $r \in \mathbb{R}$, then there exists $n \in \mathbb{N}$ such that $r < n$.

- Suppose that there is an $r \in \mathbb{R}$ such that $r > n$ for every $n \in \mathbb{N}$.
- Then the set $\mathbb{N}$ is a nonempty subset of $\mathbb{R}$ with an upper bound, so by the Completeness Axiom, $\mathbb{N}$ has least upper bound M.
- Then $M - 1 < M$, so $M - 1$ is not an upper bound for $\mathbb{N}$.
- Thus, there is a $k \in \mathbb{N}$ with the property that $k > M - 1$.
- But then $k + 1$ is also in $\mathbb{N}$, yet $k + 1 > (M - 1) + 1 = M$ where M is an upper bound for $\mathbb{N}$.
- This is a contradiction since no element of a set can be greater than an upper bound for that set.
- Therefore, the assumption that $r > n$ for every $n \in \mathbb{N}$ must be false, and for every $r \in \mathbb{R}$ there must be at least one $n \in \mathbb{N}$ with $n > r$.