behave as x tends to $+\infty$? [If $\alpha \neq \beta$ the behaviour of

$$f(x) = x^{\alpha-\beta} (\log x)^{\alpha'-\beta'} (\log\log x)^{\alpha''-\beta''}$$

is dominated by that of $x^{\alpha-\beta}$. If $\alpha=\beta$ the power of x disappears and the behaviour of $f(x)$ is dominated by that of $(\log x)^{\alpha'-\beta'}$, unless $\alpha'=\beta'$, when it is dominated by that of $(\log\log x)^{\alpha''-\beta''}$. Thus $f(x) \to +\infty$ if $\alpha>\beta$, or $\alpha=\beta$, $\alpha'>\beta'$, or $\alpha=\beta$, $\alpha'=\beta'$, $\alpha''>\beta''$, and $f(x)\to 0$ if $\alpha<\beta$, or $\alpha=\beta$, $\alpha'<\beta'$, or $\alpha=\beta$, $\alpha'=\beta'$, $\alpha''<\beta''$.]

5. Arrange the functions $x/\sqrt{(\log x)}$, $x\sqrt{(\log x)}/\log\log x$, $x\log\log x/\sqrt{(\log x)}$, $(x\log\log\log x)/\sqrt{(\log\log x)}$ according to the rapidity with which they tend to $+\infty$ with x.

6. Arrange

$$\log\log x/(x\log x),\quad (\log x)/x,\quad x\log\log x/\sqrt{(x^2+1)},\quad \{\sqrt{(x+1)}\}/x(\log x)^2$$

according to the rapidity with which they tend to zero as x tends to $+\infty$.

7. Arrange

$$x\log\log(1/x),\quad \sqrt{[x/\{\log(1/x)\}^3]},\quad \sqrt{\{x\sin x\log(1/x)\}},\quad (1-\cos x)\log(1/x)$$

according to the rapidity with which they tend to zero as $x\to+0$.

8. Show that

$$D_x \log\log x = 1/(x\log x),\quad D_x \log\log\log x = 1/(x\log x\log\log x),$$

and so on.

9. Show that

$$D_x (\log x)^\alpha = \alpha/\{x(\log x)^{1-\alpha}\},\quad D_x(\log\log x)^\alpha = \alpha/\{x\log x(\log\log x)^{1-\alpha}\},$$

and so on.

185. The number e. We shall now introduce a number, usually denoted by e, which is of immense importance in higher mathematics. It is, like π, one of the fundamental constants which perpetually occur in analysis.

We define e as *the number whose logarithm is* 1. In other words e is defined by the equation

$$1 = \int_1^e \frac{dt}{t}.$$

Since $\log x$ continually increases with x, it can only pass *once* through the value 1. Hence our definition does in fact define one definite number.

Example. Prove that $2<e<3$. [In the first place it is evident that

$$\int_1^2 \frac{dt}{t} < 1,$$

and so $2 < e$. Also

$$\int_1^3 \frac{dt}{t} = \left(\int_1^{5/4} + \int_{5/4}^{3/2} + \int_{3/2}^{7/4} + \int_{7/4}^{2} + \dots + \int_{11/4}^{3}\right)\frac{dt}{t}$$
$$> \tfrac{1}{4}\{\tfrac{4}{5}+\tfrac{2}{3}+\tfrac{4}{7}+\tfrac{1}{2}+\tfrac{4}{9}+\tfrac{2}{5}+\tfrac{4}{11}+\tfrac{1}{3}\} > 1,$$

so that $e < 3$.]

Now $\log(xy) = \log x + \log y$ and in particular

$$\log x^2 = 2\log x,\ \log x^3 = 3\log x,\ \dots,\ \log x^n = n\log x,$$

where n is any positive integer. Hence

$$\log e^n = n\log e = n.$$

Again, if p and q are any positive integers, and $e^{p/q}$ denotes the positive qth-root of e^p, we have

$$p = \log e^p = \log(e^{p/q})^q = q\log e^{p/q},$$

so that $\log(e^{p/q}) = p/q$. Thus if y has any positive rational value and e^y denotes the positive yth-power of e, we have

$$\log e^y = y \quad \dots\dots\dots\dots(1),$$

and $\log e^{-y} = -\log e^y = -y$. Hence the equation (1) is true for all rational values of y, positive or negative. In other words the equations

$$y = \log x, \quad x = e^y \dots\dots\dots\dots(2)$$

are consequences of one another so long as y is rational and e^y has its positive value. At present we have not given any definition of a power such as e^y in which the index is irrational, and the function e^y is defined for rational values of y only.

186. The exponential function e^y. We now define the *exponential function* e^y for all real values of y as the inverse of the logarithmic function. In other words, if $y = \log x$, we write

$$x = e^y.$$

We saw that, as x varies from 0 towards $+\infty$, y increases steadily (in the stricter sense) from $-\infty$ towards $+\infty$. To one value of x corresponds one value of y and conversely. Also y is a continuous function of x, and it follows from § 88 that x is likewise a continuous function of y.

A direct proof of the continuity of the exponential function is easily given. For if $x = e^y$ and $x + \xi = e^{y+\eta}$, it is clear that

$$\eta = \int_x^{x+\xi} \frac{dt}{t}.$$

Thus $|\eta|$ is greater than $\xi/(x+\xi)$ if $\xi > 0$, and than $|\xi|/x$ if $\xi < 0$; and if η is very small ξ must also be very small.

Thus e^y is a continuous function of y which increases steadily from 0 towards $+\infty$ as y increases from $-\infty$ towards $+\infty$. Moreover, by the results of § 185, e^y is the positive yth-power of the number e, according to the elementary definitions, whenever y is a rational number. In particular $e^y = 1$ when $y = 0$. The general form of the graph of e^y is therefore as shown in Fig. 68. It is to be observed that e^y is positive for all values of y.

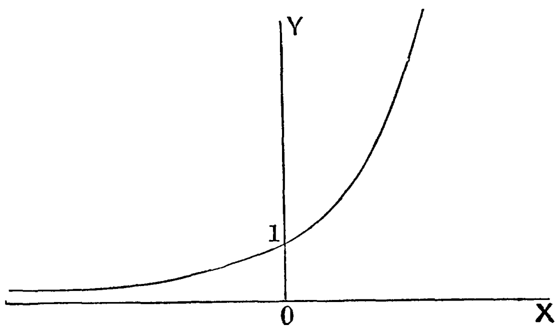

FIG. 68.

187. The principal properties of e^y. (1) If $x = e^y$, so that $y = \log x$, we have $dy/dx = 1/x$ and

$$dx/dy = x = e^y.$$

Thus *the derivative of the exponential function is equal to the function itself.* In other words the exponential function is a function whose rate of increase is always equal to its own value. More generally, if $y = e^{ax}$ then $dy/dx = ae^{ax}$.

(2) *The exponential function satisfies the functional equation*

$$f(x+y) = f(x)f(y).$$

This is evident if x and y are rational, by the ordinary rules of indices. If x or y, or both, are irrational we can choose two series of rational numbers $x_1, x_2, \dots, x_n, \dots$; $y_1, y_2, \dots, y_n, \dots$, such that $\lim x_n = x$, $\lim y_n = y$. Then, since the exponential function is continuous,

$$e^x \times e^y = \lim e^{x_n} \times \lim e^{y_n} = \lim e^{x_n + y_n} = e^{x+y}.$$

In particular $e^x \times e^{-x} = e^0 = 1$, or $e^{-x} = 1/e^x$.

Or we may deduce the functional equation satisfied by e^y from that satisfied by $\log x$. For if $y_1 = \log x_1$, $y_2 = \log x_2$, so that $x_1 = e^{y_1}$, $x_2 = e^{y_2}$, we have $y_1 + y_2 = \log x_1 + \log x_2 = \log(x_1 x_2)$ and

$$e^{y_1+y_2} = e^{\log(x_1 x_2)} = x_1 x_2 = e^{y_1} \times e^{y_2}.$$

Examples LXXXVII. 1. If $dx/dy = x$ then $x = Ke^y$, where K is a constant.

2. There is no solution of the equation $f(x+y) = f(x)f(y)$ fundamentally distinct from $f(y) = e^y$. [For, differentiating the equation with respect to x and y in turn, we obtain

$$f'(x+y) = f'(x)f(y), \quad f'(x+y) = f(x)f'(y)$$

and so $f'(x)/f(x) = f'(y)/f(y)$, and therefore each is constant. Thus if $z = f(y)$, $dz/dy = z/A$, or

$$y = A\int \frac{dz}{z} = A\log z + B,$$

A and B being constants; so that $z = e^{(y-B)/A}$.]

3. Prove that $(e^{ax} - 1)/x \to a$ as $x \to 0$. [Applying the Mean Value Theorem, we obtain $e^{ax} - 1 = ae^{a\xi}$ where $0 < \xi < x$.]

188. (3) *The function e^y tends to $+\infty$ with y more rapidly than any power of y, or*

$$\lim y^\alpha/e^y = \lim e^{-y}y^\alpha = 0$$

as $y \to +\infty$, for all values of α, however great.

We saw that $(\log x)/x^\beta \to 0$ as $x \to +\infty$, for any positive value of β, however small. Hence, if $\alpha = 1/\beta$, $(\log x)^\alpha/x \to 0$ for any value of α, however large. The result follows by putting $x = e^y$.

From this result it follows that we can construct a 'scale of infinity' similar to that constructed in § 184, but extended in the opposite direction—i.e. a scale of functions which tend to $+\infty$ with x more and more rapidly. This scale is

$$x,\ x^2,\ x^3,\ \dots\quad e^x,\ e^{2x},\ \dots\quad e^{x^2},\ \dots,\ e^{x^3},\ \dots,\ e^{e^x},\ \dots\quad e^{e^{e^x}},\ \dots,$$

where of course $e^{x^2}, \dots, e^{e^x}, \dots$ denote $e^{(x^2)}, \dots, e^{(e^x)}, \dots$.

The reader should try to apply the remarks made in § 184 and Exs. LXXXVI, about the logarithmic scale, to this 'exponential scale' also. The two scales may of course (if the order of one is reversed) be combined into one scale

$$\dots \log\log x,\ \dots\quad \log x,\ \dots\quad x,\ \dots\quad e^x,\ \dots\quad e^{e^x},\ \dots$$

189. The general power a^x. The function a^x has been defined only for rational values of x, except in the particular case when $a = e$. When a is rational and positive, the positive value of the power a^x is given by the equations

$$a^x = (e^{\log a})^x = e^{x\log a}.$$

We take this as our *definition* of a^x when x is irrational. Thus $10^{\sqrt{2}} = e^{\sqrt{2}\cdot\log 10}$. It is to be observed that a^x, when x is irrational, is defined only for positive values of a, and is itself essentially positive. The reader will find no difficulty in verifying the following statements.

(1) Whatever value a may have, $a^x \times a^y = a^{x+y}$ and $(a^x)^y = a^{xy}$. In other words the laws of indices hold for irrational no less than for rational indices.

(2) If $a > 1$, $a^x = e^{x \log a} = e^{\alpha x}$, where α is positive. The graph of a^x is in this case similar to that of e^x, and $a^x \to +\infty$ with x, more rapidly than any power of x. For if $\alpha > 0$ and $\alpha x = y$, then $x^m e^{-\alpha x} = (y/\alpha)^m e^{-y} \to 0$ as x and y tend to $+\infty$.

If $a < 1$, $a^x = e^{x \log a} = e^{-\alpha x}$ where α is positive. The graph of a^x is then similar in shape to that of e^x, but reversed as regards right and left, and $a^x \to 0$ as $x \to +\infty$, more rapidly than any power of $1/x$.

(3) a^x is a continuous function of x, and $D_x a^x = a^x \log a$.

(4) a^x is also a continuous function of a, and $D_a a^x = x a^{x-1}$.

(5) $(a^x - 1)/x \to \log a$ as $x \to 0$. This of course is a mere corollary from the fact that $D_x a^x = a^x \log a$, but the particular form of the result is often useful; it is of course equivalent to the result (Ex. LXXXVII. 3) that $(e^{\alpha x} - 1)/x \to \alpha$ as $x \to 0$.

In the course of the preceding chapters a great many results involving the function a^x have been stated with the limitation that x is rational. The definition which we have now given, and the theorems proved above, enable us to remove this restriction.

190. The representation of e^y as a limit. In Ch. IV, § 67, we proved that $\{1 + (1/n)\}^n$ tends, as $n \to \infty$, to a limit which we provisionally denoted by e. We shall now identify this limit with the number e of the preceding sections. We can however establish a more general result, viz. that expressed by the equations

$$\lim_{n \to \infty} \left(1 + \frac{y}{n}\right)^n = \lim_{n \to \infty} \left(1 - \frac{y}{n}\right)^{-n} = e^y \ldots\ldots\ldots\ldots (1).$$

As the result is of very great importance, we shall indicate alternative lines of proof.

(1) Since $D_x \log(1 + yx) = y/(1 + yx)$, it follows that

$$\lim \{\log(1 + yh)\}/h = y,$$

as $h \to 0$; or, putting $h = 1/\xi$,

$$\lim [\xi \log\{1 + (y/\xi)\}] = y$$