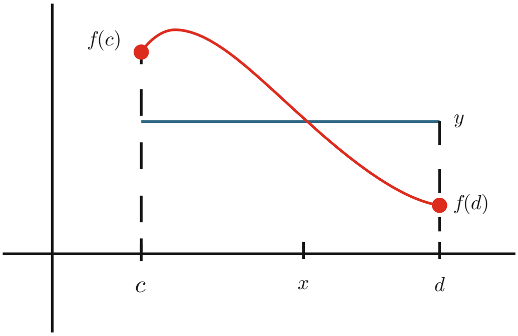
PROOF (Extreme Value Theorem): A function continuous on a closed bounded interval obtains its maximum value and its minimum value at some points in the interval.

- Let $a \le b$ be real numbers, and let f be a function continuous on the interval $[a, b]$.
- The set $B = \{f(x) \mid x \in [a, b]\}$ is not empty because it contains $f(a)$, and B is bounded above because all functions continuous on a closed bounded interval are bounded.
- Let M be the least upper bound of set B.
- Assume that for all $x \in [a, b], f(x) \ne M$.
- The function $M - f(x)$ is continuous on $[a, b]$ and is never equal to 0. Hence, $M - f(x) > 0$ on $[a, b]$.
- It follows that the function $\frac{1}{M-f(x)}$ is continuous on $[a, b]$.
- Because all functions continuous on a closed bounded interval are bounded, there is a real number $K > 0$ such that $\frac{1}{M-f(x)} \le K$ on $[a, b]$.
- But then $M - f(x) \ge \frac{1}{K}$ on $[a, b]$, so $f(x) \le M - \frac{1}{K}$ on $[a, b]$.
- Since $K > 0$, the set B is bounded above by $M - \frac{1}{K} < M$. This means that $M - \frac{1}{K}$ is an upper bound for B which contradicts the fact that M was the least upper bound of B.
- Therefore, the assumption that f was never equal to M is false, and there must be a value $x \in [a, b]$ such that $f(x) = M$.
- Applying the proceeding argument to the function $-f$, which is also continuous on $[a, b]$, shows that there is an $x \in [a, b]$ such that $-f(x)$ is equal to the maximum of $-f$ on $[a, b]$. But then $f(x)$ is the minimum value of f on $[a, b]$. This completes the proof of the theorem.

4.7.3 The Intermediate Value Property

Suppose the function f is defined on an interval containing c and d, and the graph of f passes through the points $(c, f(c))$ and $(d, f(d))$. It might be that the graph of the function passes through every value of y between $f(c)$ and $f(d)$ as it moves between the points $(c, f(c))$ and $(d, f(d))$ as shown in the figure (Fig. 4.9). For example, the function $f(x) = 2x^2 - 3$ is defined for all real numbers with $f(1) = -1$ and $f(2) = 5$. For each y between -1 and 5, the value $x = \sqrt{\frac{y+3}{2}}$ lies between 1 and 2 and $f(x) = y$. Formally, a function defined on an interval $[a, b]$ is said to have the **intermediate value property** on that interval if for each choice of c and d with $a \le c \le d \le b$ and each y between $f(c)$ and $f(d)$, there is an $x \in [c, d]$ such that $f(x) = y$. The Intermediate Value Theorem states that any function continuous on an interval has the intermediate value property there. If you consider the intuitive notion of continuity where you say that f is continuous on $[a, b]$ if you can draw the graph of

Fig. 4.9 f passing through each y between $f(c)$ and $f(d)$



f without lifting your pencil from the paper, then this intermediate value property becomes clear because in going from $f(c)$ to $f(d)$, your pencil will necessarily cross over all the y values between $f(c)$ and $f(d)$.

To prove the Intermediate Value Theorem you would begin by setting the context by introducing a function f continuous on an interval $[a, b]$ and points c and d with $a \leq c \leq d \leq b$. Then you would select an arbitrary y between $f(c)$ and $f(d)$. The proof would have to demonstrate the existence of an x between c and d with $f(x) = y$. How is this to be done? As with many other proofs in Analysis, one shows the existence of a real number by constructing a set for which that number is a least upper bound. Consider, for example, the case where $f(c) < y < f(d)$. You could construct the set $S = \{x \in [c, d] \mid f(x) \leq y\}$. This set is not an empty set because $c \in S$, and S is certainly bounded above by d. Thus, the Completeness Axiom says that the set has a least upper bound, s. Now you can refer to the continuity of f which will show that if $f(s) < y$, then there is a $\delta > 0$ such that $|x - s| < \delta$ implies that $f(x) < y$ showing that there are values greater than s for which $f(x) < y$ contradicting the fact that s is an upper bound of S. If $f(s) > y$, then there is a $\delta > 0$ such that $|x - s| < \delta$ implies that $f(x) > y$ showing that $s - \delta < s$ is an upper bound for S contradicting the fact that s is the least upper bound of S. The only remaining conclusion is that $f(s) = y$ which provides the needed example, $x = s$, needed to prove the theorem.

Note that the above argument did not cover the general case where $f(c)$ and $f(d)$ can be in any order. The argument so far only covers the specific case where $f(c) < f(d)$. So is there more proof to write? It is easy to see that the case $f(c) > f(d)$ can be proved with an argument virtually identical to the one given above by changing the sense of some of the inequalities. The case of $f(c) = f(d)$ is even easier because the only possible y between $f(c)$ and $f(d)$ is $f(c)$, so the value $x = c$ gives the needed $f(x) = y$. Thus, giving the argument for $f(c) < f(d)$ essentially covers all the needed cases, and it would be very easy for the reader to add the needed arguments to complete the proof for the missing cases. In this situation it is common for the proof to cover only the specific condition $f(c) < f(d)$ and introduce it with the phrase *without loss of generality*. In this case the phrase means that although the following assumption looks like it only covers some of the necessary cases, in order to make the argument completely general, the omitted cases are either very easy or virtually identical to the case being considered. With this in mind, the following is a proof of the Intermediate Value Theorem.

PROOF (Intermediate Value Theorem): Let the function f be continuous on the interval $[a, b]$ containing c and d. If y is any value between $f(c)$ and $f(d)$, then there exists x between c and d such that $f(x) = y$.

- Let f be a function continuous on $[a, b]$, and let c and d be in $[a, b]$.
- Let y be any value between $f(c)$ and $f(d)$.
- Without loss of generality, assume that $c \leq d$ and $f(c) \leq y \leq f(d)$.
- Let set $S = \{x \in [c, d] \mid f(x) \leq y\}$.
- S is not empty because $f(c) \leq y$ implying $c \in S$.
- S is bounded above by d.
- By the Completeness Axiom S has a least upper bound s which will be an element of $[a, b]$.
- If $f(s) < y$, then by the continuity of f, there is a $\delta > 0$ such that if $x \in [a, b]$ with $|x - s| < \delta$, then $|f(x) - f(s)| < \frac{y-f(s)}{2}$, and, in particular, $f(x) < y$. This shows that there is an $x > s$ with $f(x) < y$, so $x \in S$ contradicting the fact that s is an upper bound of S.
- If $f(s) > y$, then by the continuity of f, there is a $\delta > 0$ such that if $x \in [a, b]$ with $|x - s| < \delta$, then $|f(x) - f(s)| < \frac{f(s)-y}{2}$, and, in particular, $f(x) > y$. This shows that for all x between $s - \delta$ and s that $f(x) > y$, so $s - \delta$ is an upper bound of S contradicting the fact that s is the least upper bound of S.
- It follows that $f(s)$ must equal y which completes the proof of the theorem.

In the above proof the steps which begin "If $f(x) < y$" and "If $f(x) > y$" are written in exactly the same style using almost identical words. If you were writing a short story, you would avoid writing in this style because it might sound monotonous to the reader. In creative writing, you would want to be more creative, and you would reach for your thesaurus to find alternate words to enhance your writing. But in a mathematical proof, using such parallel construction of sentences actually makes the proof easier to read. A reader only needs to parse the first of the two steps in order to have a good idea of what is going to be done in the second of the two steps. This gives the reader a head start on processing the second step. What is passed off as boring in creative writing can be applauded in the writing of proofs because of the way it simplifies the understanding. In fact, one often begins the second of two such steps with the word *similarly* to indicate that the argument to follow looks a lot like the one just completed, again alerting the reader to the parallel construction.

The Intermediate Value Theorem says that functions continuous on an interval have the intermediate value property there. But a function need not be continuous for it to have the intermediate value property. Clearly, if a function has a jump discontinuity at a point a, that is, if $\lim_{x \to a^-} f(x)$ and $\lim_{x \to a^+} f(x)$ both exist but are different as shown in Fig. 4.10, then there could well be values of y that the function misses as it passes from $(c, f(c))$ to $(d, f(d))$.

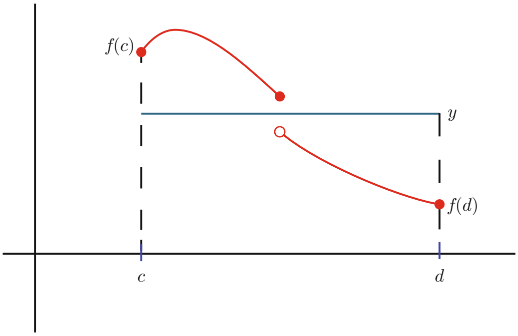


Fig. 4.10 A function with a jump discontinuity

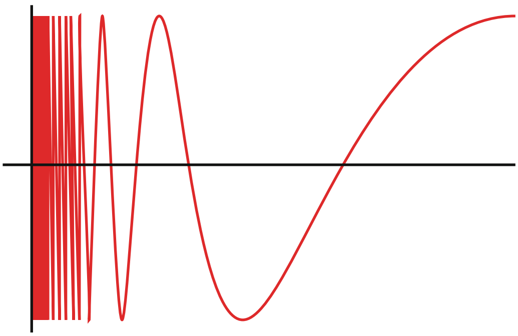


Fig. 4.11 Graph of $\sin \frac{1}{x}$

For a discontinuous function to have the intermediate value property, the function must necessarily oscillate wildly (Fig. 4.11). A typical example is the function $f(x) = \left\{ \begin{array}{ll} \sin \frac{1}{x} & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{array} \right\}$.

4.7.4 Exercises

Write proofs for each of the following statements. Each statement can be proved using one or more of the theorems in this section.

1. Let $A \subseteq \mathbb{R}$ be a bounded set, and let f be a function defined on A. If f is unbounded on A, then for every $\epsilon > 0$, there exists a and b in $\mathbb{R}$ with $b - a < \epsilon$ such that f is unbounded on $A \cap (a, b)$.
2. If $a < b$ and f is a continuous function on $[a, b]$ with $f(a) = f(b)$, then there is a $c \in (a, b)$ such that f obtains an extreme value (either a minimum or maximum) at c.
3. Suppose that f is a continuous function defined on $\mathbb{R}$ such that $\lim_{x \to -\infty} f(x) = \lim_{x \to \infty} f(x) = \infty$. Then f obtains its minimum value for some $x \in \mathbb{R}$.

4. If p is an odd degree polynomial with real coefficients, then p has at least one real root.
5. Suppose that a plane contains be a polygon G and a line L. Then there is a line L' in the plane parallel to L such that exactly half the area of G lies on each side of L'.
6. There is a value of x between 0 and 1 such that x^2 equals $\sqrt{\dfrac{1}{1+x^2}}$.

4.8 Discontinuity

In Calculus students learn about a great many continuous functions. These include the elementary functions: polynomials, rational functions, algebraic functions, exponential functions, logarithmic functions, and circular and hyperbolic trigonometric functions and their inverses. How badly can a function be discontinuous? A function can be discontinuous at a single point such as the *signum* or *sign* function $\operatorname{sgn}(x) = \left\{ \begin{array}{ll} -1 & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ 1 & \text{if } x > 0 \end{array} \right\}$ or at a sequence of points such as the *floor* or *greatest integer* function $\lfloor x \rfloor = n$ if n is the integer satisfying $n \le x < n + 1$ (Fig. 4.12). A function can be discontinuous at a sequence of points that converge such as with $f(x) = \left\{ \begin{array}{ll} \frac{1}{n} & \text{if } \frac{1}{n+1} < x \le \frac{1}{n}, \text{ for positive integer } n \\ 0 & \text{otherwise} \end{array} \right\}$. This function is discontinuous at each $x = \frac{1}{n}$ for positive integers n, but it is continuous everywhere else including at $x = 0$ (Fig. 4.13). A function can be discontinuous at every x such as with $f(x) = \left\{ \begin{array}{l} 0 \text{ if } x \text{ is rational} \\ 1 \text{ if } x \text{ is irrational} \end{array} \right\}$.
But one of the most surprising examples is the following often called Thomae's function but also known as the popcorn function, the raindrop function,

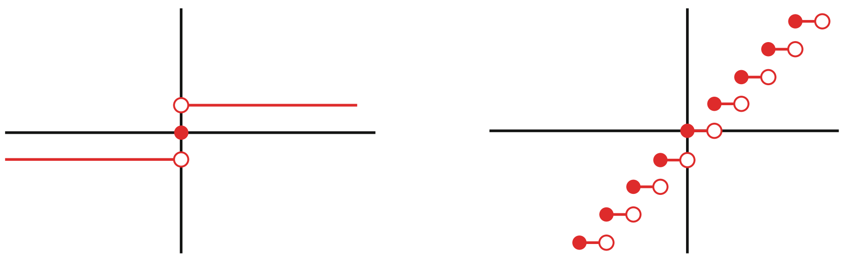


Fig. 4.12 Graphs of $\operatorname{sgn}(x)$ and $\lfloor x \rfloor$

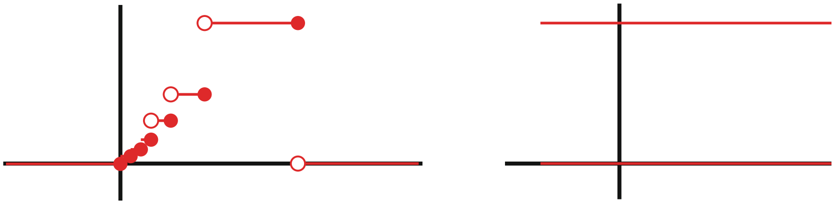
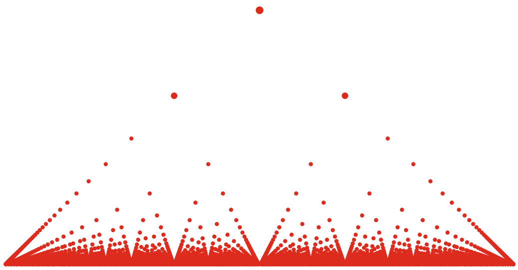


Fig. 4.13 Graphs of functions with discontinuities

Fig. 4.14 Graph of Thomae's function



or the modified Dirichlet function. It is defined on the interval $(0, 1)$ by $f(x) = \left\{ \begin{array}{l} \frac{1}{n} \text{ if } x \text{ is rational written in lowest terms as } \frac{m}{n} \\ 0 \text{ if } x \text{ is irrational} \end{array} \right\}$. Its graph is shown in Fig. 4.14. It is not hard to see that this function is discontinuous at each rational number $\frac{m}{n} \in (0, 1)$. Indeed if $\frac{m}{n}$ is in lowest terms, then $f(\frac{m}{n}) = \frac{1}{n}$. If ϵ is set at $\frac{1}{2n}$, then for every $\delta > 0$ there will be irrational numbers $x \in (0, 1)$ satisfying $\left|x - \frac{m}{n}\right| < \delta$ for which $\left|f(x) - f\left(\frac{m}{n}\right)\right| = |0 - \frac{1}{n}| > \epsilon$. On the other hand, at each irrational number a in $(0, 1)$, the function is continuous. To see this, given an $\epsilon > 0$, notice that there are only finitely many rational numbers $r \in (0, 1)$ such that $f(r) \geq \epsilon$. If there are such rational numbers, there is one, r', closest to a, so choose $\delta = |r' - a|$. If there are no such rational numbers, you can choose $\delta = 1$. In either case, for all $x \in (0, 1)$ with $|x - a| < \delta$, it follows that $|f(x) - f(a)| < \epsilon$, showing that f is continuous at a.

Chapter 5
Derivatives

5.1 The Definition of Derivative

Anybody who was even half paying attention in their first course in Calculus got the strong impression that the differentiation of functions has an enormous number of applications. Not only does it provide a great tool for understanding the behavior of functions, but it also has applications to a very wide range of other fields, most notably Physics, Engineering, Chemistry, Biology, and Economics. In particular, being able to use the derivative to determine where a function is increasing and decreasing in itself justifies this reputation. Merely knowing the average rate of change of a function over an interval is valuable. But the limit concept allows you to refine this idea to get the instantaneous rate of change of the function at a point. This allows for more precise information about the function as well as providing what is often a simpler expression than that of the average rate of change from which it is derived. This chapter will discuss the theorems needed to calculate derivatives efficiently as well as theorems highlighting some of the important properties and applications of the derivative.

Let f be a function defined on an open interval containing the point a. Then for values of x near but not equal to a one can calculate the slope of the **secant line** passing through the two points on the graph of the function $\big(a, f(a)\big)$ and $\big(x, f(x)\big)$. As shown in Fig. 5.1, the slope of this secant line is given by the **difference quotient** $\frac{f(x)-f(a)}{x-a}$. If f is continuous, as x approaches a, the point $\big(x, f(x)\big)$ approaches the point $\big(a, f(a)\big)$, and the secant line may approach a **tangent line**, the line that passes through $\big(a, f(a)\big)$ and most closely approximates the graph of the function near a (Fig. 5.2). The derivative of f at a is the **slope of this tangent line**. More formally, if a is an accumulation point of the domain of the function f, and f is defined at a, then the **derivative of f at a** is $f'(a) = \lim\limits_{x \to a} \frac{f(x)-f(a)}{x-a}$. The derivative is said to exist if this limit exists. When the limit exists, f is said to be **differentiable at a**. Equivalently, the limit can be written $f'(a) = \lim\limits_{h \to 0} \frac{f(a+h)-f(a)}{h}$.

J.M. Kane, *Writing Proofs in Analysis*, DOI 10.1007/978-3-319-30967-5_5

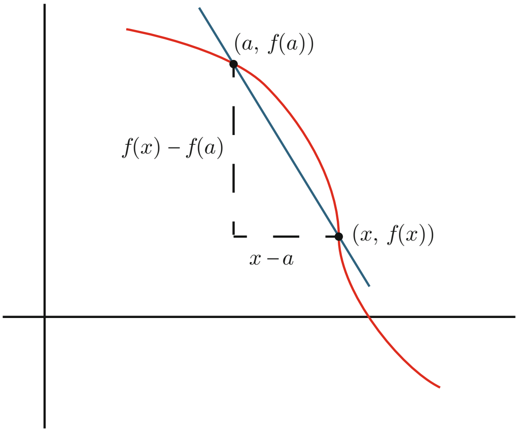


Fig. 5.1 Slope of a Secant Line

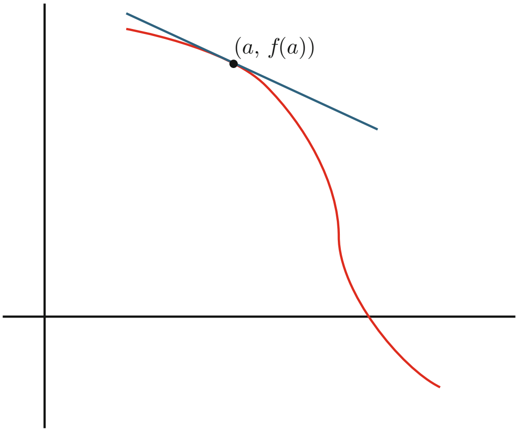


Fig. 5.2 Tangent Line

5.2 Differentiation and Continuity

The first important consequence of the definition of the derivative is that if a function f has a derivative at a point, then f is also continuous at that point. As part of the definition of derivative, f needs to be defined at the point a for it to have a derivative at a. It remains to show that $\lim_{x\to a} f(x) = f(a)$ whenever the limit $\lim_{x\to a} \frac{f(x)-f(a)}{x-a}$ exists. For this difference quotient to have a finite limit when the denominator is clearly approaching 0, the numerator must also be approaching 0. This last statement is intuitively true, so you would hope that it has an easy justification. Consider what sort of algebraic operations you could apply to the difference quotient $\frac{f(x)-f(a)}{x-a}$ in order to produce the numerator $f(x) - f(a)$. It should be clear that if the difference quotient is multiplied by $x-a$, the product will be the desired difference $f(x)-f(a)$. This suggests the method that works in the following simple proof.

PROOF: If the function *f* has a derivative at a point *a*, then *f* is continuous at *a*.

- Suppose that f has a derivative at the point a.
- It follows from the definition of derivative that f is defined at a, and that a is an accumulation point of the domain of f.
- Also from the definition of derivative it follows that $f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}$ exists.
- Then $\lim_{x\to a} f(x) - f(a) = \lim_{x\to a} (x-a)\frac{f(x)-f(a)}{x-a} = \left[\lim_{x\to a} x-a\right] \cdot \lim_{x\to a} \frac{f(x)-f(a)}{x-a} = 0 \cdot f'(a) = 0$.
- Thus, $f(x)$ is both defined at $x = a$, and $\lim_{x\to a} f(x) - f(a) = 0$, or $\lim_{x\to a} f(x) = f(a)$.
- It follows that f is continuous at $x = a$.

So, f differentiable at a implies that f is continuous at a. Is the converse true? You should know several counterexamples that show that the converse is false, that is, there are functions f continuous at a point a that are not differentiable at a. First of all, f can be continuous at a where a is an isolated point of the domain of f, and at such points, the derivative of f is not defined. But even if f is continuous for all real numbers, f need not have a derivative at a particular a. The best known example is the absolute value function, $f(x) = |x|$, which is continuous for all real numbers but fails to have a derivative at $x = 0$. This is because the difference quotient $\frac{f(x)-f(0)}{x-0}$ is equal to 1 for all $x > 0$ and -1 for all $x < 0$, so the limit of the difference quotient does not exist at $x = 0$. Of course, the absolute value function has a derivative at all $x \neq 0$. There is a well-known example known as the Weierstrauss function that is continuous for real numbers x but does not have a derivative at any point.

5.3 Calculating Derivatives

The proof that a function f has a particular derivative at a point a is just a proof about the limit of a difference quotient, and as such, is no different than a proof of any other limit. On the other hand, there are some similarities among the proofs of derivatives, so it is worth working through a few examples. The key observation is that whenever you need to calculate a derivative directly from the definition, you must calculate the limit of a difference quotient which, by design, is a fraction whose numerator and denominator are both approaching zero. In such a case, one would expect to be able to perform some algebraic manipulation that would result in the $x - a$ expression in the denominator canceling with an equivalent factor in the numerator. This allows you to use other limit theorems to complete the evaluation.

For example, consider the function $f(x) = 3x^2 - 8x$. To calculate the derivative of f at $a = 4$, one needs to evaluate the limit

$$\lim_{x\to4}\frac{f(x)-f(4)}{x-4} = \lim_{x\to4}\frac{\left[3x^2-8x\right]-\left[3\cdot4^2-8\cdot4\right]}{x-4} = \lim_{x\to4}\frac{3x^2-8x-16}{x-4}$$
$$= \lim_{x\to4}\frac{(3x+4)(x-4)}{x-4} = \lim_{x\to4} 3x+4 = 16.$$

Since each step of this derivation follows either from rules of algebra or from the theorems about calculating the limits of various arithmetic combinations of functions, the calculation given is a complete proof that the derivate of f at $x = 4$ is 16.

In a more general setting, consider proving that the derivative of $f(x) = 5x^4$ at the point $x = a$ is $f'(a) = 20a^3$. Here you would calculate

$$\lim_{x\to a}\frac{f(x)-f(a)}{x-a} = \lim_{x\to a}\frac{5x^4-5a^4}{x-a} = \lim_{x\to a}\frac{5(x-a)\left(x^3+x^2a+xa^2+a^3\right)}{x-a}$$
$$= \lim_{x\to a}5(x^3+x^2a+xa^2+a^3) = 5(a^3+a^2a+aa^2+a^3) = 20a^3.$$

Again, finding a factor of $x-a$ in the numerator of the difference quotient is the key to evaluating the needed limit.

5.4 The Arithmetic of Derivatives

One quickly learns in Calculus that although the derivative is defined as a limit of a difference quotient, there is a small collection of algorithms that reduce the finding of the derivative of any combination of elementary functions to a fairly mechanical exercise. The algorithms show you how to take the derivatives of the sum, difference, product, and quotient of two differentiable functions as well as a constant multiple of a differentiable function, the inverse of a differentiable function, and the composition of two differentiable functions. Those rules along with the knowledge of how to differentiate the elementary functions, x^n, a^x, $\log_a x$, $\sin x$, and $\cos x$ give you all the tools necessary to differentiate virtually any function you are likely to see in a lifetime of applications. This and the next sections discusses the proofs of the theorems that provide these needed algorithms.

The simplest of these results is the theorem that states that if f is a function differentiable at a and c is any constant, then the function cf is also differentiable at a with $(cf)'(a) = cf'(a)$. In the proof of this theorem, you would assume that $f'(a)$ exists. That provides for you the limit $\lim\limits_{x\to a}\frac{f(x)-f(a)}{x-a} = f'(a)$. Since the limit needed to show that $(cf)'(a) = cf'(a)$ is just a multiple of a known limit, the needed result follows immediately from the fact that the limit of a constant times a function is the constant times the limit of the function.

PROOF: If $f'(a)$ exists, and c is a constant, then $(cf)'(a) = cf'(a)$.

- Suppose that f has a derivative at the point a.
- From the definition of derivative $f'(a) = \lim_{x \to a} \frac{f(x)-f(a)}{x-a}$.
- Then $(cf)'(a) = \lim_{x \to a} \frac{cf(x)-cf(a)}{x-a} = \lim_{x \to a} c \cdot \frac{f(x)-f(a)}{x-a} = c \lim_{x \to a} \frac{f(x)-f(a)}{x-a} = cf'(a)$.
- Thus, $(cf)'(a) = cf'(a)$.

To show that the derivative of the sum or difference of two differentiable functions is the sum or difference of their derivatives, one is faced with finding the limit of a difference quotient which can easily be written as the sum or difference of two difference quotients whose limits are already known. Thus, if f and g are two functions defined on the same domain and both differentiable at a, calculating the derivative of $f + g$ at a requires the limit

$$\lim_{x \to a} \frac{(f+g)(x) - (f+g)(a)}{x-a} = \lim_{x \to a} \frac{\big(f(x) - f(a)\big) + \big(g(x) - g(a)\big)}{x-a}$$

$$= \lim_{x \to a} \frac{f(x) - f(a)}{x-a} + \lim_{x \to a} \frac{g(x) - g(a)}{x-a} = f'(a) + g'(a)$$

as needed.

PROOF: Let f and g be functions defined on a common domain, and let f and g both be differentiable at a. Then $(f+g)'(a) = f'(a) + g'(a)$ and $(f-g)'(a) = f'(a) - g'(a)$.

- Suppose that f and g are functions defined on a common domain, and that f and g are both differentiable at a.
- From the definition of derivative $f'(a) = \lim_{x \to a} \frac{f(x)-f(a)}{x-a}$ and $g'(a) = \lim_{x \to a} \frac{g(x)-g(a)}{x-a}$.
- Then $(f+g)'(a) = \lim_{x \to a} \frac{(f+g)(x)-(f+g)(a)}{x-a} = \lim_{x \to a} \frac{\big(f(x)-f(a)\big)+\big(g(x)-g(a)\big)}{x-a} = \lim_{x \to a} \frac{f(x)-f(a)}{x-a} + \lim_{x \to a} \frac{g(x)-g(a)}{x-a} = f'(a) + g'(a)$.
- Thus, $(f+g)'(a) = f'(a) + g'(a)$.
- Because $(f-g)(x) = f(x) - g(x) = f(x) + (-1)g(x)$, and the derivative of $(-1)g(x)$ is -1 times the derivative of $g(x)$, it follows that $(f-g)'(a) = \big(f + (-1)g\big)(a) = f'(a) + (-1)g'(a) = f'(a) - g'(a)$ completing the proof of the theorem.

Why does the first step in this proof make the assumption that f and g are defined on the same domain? This is to avoid the embarrassing situation that the intersection of the domains of f and g isolates the point a. For example, if f is defined for all $x \geq 1$ and g is defined for all $x \leq 1$, it could be that both $f'(1)$ and $g'(1)$ are defined, but the function $f+g$ is defined only at 1, so its derivative cannot be defined. Another example would be for f to be defined at all rational numbers, and g to be defined at all rational multiples of $\sqrt{2}$. Each function could be differentiable at each point of its domain, but $f + g$ is only defined at 0, so its derivative cannot be defined.

It is certainly worth noting here that the theorems discussed so far show that for functions f and g and constants a and b, the derivative of the linear combination of functions $a{\cdot}f(x)+b{\cdot}g(x)$ is the linear combination of the derivatives $a{\cdot}f'(x)+b{\cdot}g'(x)$. In the words of Linear Algebra, this says that the derivative is a linear operator. This fact alone has a long list of ramifications in Differential Equations and other fields.

It is important for the beginning Calculus student to learn that even though the derivative behaves in an intuitive way with respect to addition and subtraction, that this intuition ceases when discussing the derivative of a product or quotient. The proof that $(fg)' = fg' + f'g$ involves one trick reminiscent of the proof that the limit of a product is the product of the limits. That is, one adds and subtracts the same quantity so that rather than making a change in two different factors at the same time, one makes a change in one factor at a time. Indeed, the difference quotient you obtain for the function fg is

$$\begin{aligned}\frac{f(x)g(x)-f(a)g(a)}{x-a} &= \frac{f(x)g(x)-f(x)g(a)+f(x)g(a)-f(a)g(a)}{x-a} \\ &= f(x)\cdot\frac{g(x)-g(a)}{x-a} + \frac{f(x)-f(a)}{x-a}\cdot g(a).\end{aligned}$$

Taking the limits at each step produces the following proof.

PROOF (Product Rule): Let f and g be functions defined on a common domain, and let f and g both be differentiable at a. Then $(fg)'(a) = f(a)g'(a) + f'(a)g(a)$.

- Suppose that f and g are functions defined on a common domain, and that f and g are both differentiable at a.
- From the definition of derivative $f'(a) = \lim\limits_{x\to a}\frac{f(x)-f(a)}{x-a}$ and $g'(a) = \lim\limits_{x\to a}\frac{g(x)-g(a)}{x-a}$.
- Because f is differentiable at a, it is continuous at a. This implies that $\lim\limits_{x\to a} f(x) = f(a)$.
- Then $(fg)'(a) = \lim\limits_{x\to a}\frac{(fg)(x)-(fg)(a)}{x-a} = \lim\limits_{x\to a}\frac{f(x)g(x)-f(a)g(a)}{x-a} = \lim\limits_{x\to a}\frac{f(x)g(x)-f(x)g(a)+f(x)g(a)-f(a)g(a)}{x-a} = \lim\limits_{x\to a}\left(f(x)\cdot\frac{g(x)-g(a)}{x-a} + \frac{f(x)-f(a)}{x-a}\cdot g(a)\right) = \lim\limits_{x\to a} f(x)\cdot\lim\limits_{x\to a}\frac{g(x)-g(a)}{x-a} + \lim\limits_{x\to a}\frac{f(x)-f(a)}{x-a}\cdot\lim\limits_{x\to a} g(a) = f(a)g'(a) + f'(a)g(a)$.
- Thus, $(fg)'(a) = f(a)g'(a) + f'(a)g(a)$.

The proof that $\left(\frac{f}{g}\right)' = \frac{gf'-fg'}{g^2}$ involves the same strategy along with the extra assumption that $g(a) \neq 0$.