3.5.7 Cauchy Sequences

A **Cauchy sequence** is a sequence whose terms get close together. As with the definition of *limit*, the concept of “close” needs to be made precise. As with the definition of *limit*, “close” means that given any tolerance $\epsilon > 0$, one can go out far enough in the sequence to ensure that all terms of the sequence beyond that point are within ϵ of each other. Thus, a sequence is Cauchy if for every $\epsilon > 0$ there is an N such that if natural numbers m and n are both greater than N, then $|a_m - a_n| < \epsilon$.

If a sequence of real numbers converges, then the sequence is Cauchy. The proof of this fact uses a strategy employed repeatedly in Analysis, that is, *if two quantities are very close to the same value, then they must be very close to each other*. This standard technique for proving that two quantities are close to each other involves the use of the *triangle inequality*. In particular, if $\lim\limits_{j\to\infty} a_j = L$, then for every $\epsilon > 0$ there is an N such that if natural number $n > N$, then $|a_n - L| < \epsilon$. Well then, certainly if m and n are both natural numbers greater than N, then both $|a_m - L| < \epsilon$ and $|a_n - L| < \epsilon$. Adding these two inequalities together shows that $|a_m - L| + |a_n - L| < \epsilon + \epsilon$. The triangle inequality states that for any real numbers x and y, $|x| + |y| \geq |x + y|$. Thus, $2\epsilon > |a_m - L| + |a_n - L| = |a_m - L| + |L - a_n| \geq |(a_m - L) + (L - a_n)| = |a_m - a_n|$. Of course, the definition of *Cauchy sequence* requires you to show that $|a_m - a_n|$ is less than ϵ, not 2ϵ. But you have an enormous amount of flexibility when working with these types of inequalities, so you could have asked instead for an N such that for all natural numbers n greater than N, you have $|a_n - L|$ less than $\frac{\epsilon}{2}$ rather than less than ϵ. Thus, the proof could be as follows.

PROOF: Every convergent sequence is Cauchy.

- Let $<a_j>$ be a sequence of real numbers with $\lim\limits_{j\to\infty} a_j = L$.
- Let $\epsilon > 0$ be given.
- From the definition of limit, there is a number N such that for all natural numbers $j > N$, it follows that $|a_j - L| < \frac{\epsilon}{2}$.
- Then for all natural numbers m and n greater than N, $|a_m - L| < \frac{\epsilon}{2}$ and $|a_n - L| < \frac{\epsilon}{2}$, so $\epsilon = \frac{\epsilon}{2} + \frac{\epsilon}{2} > |a_m - L| + |a_n - L| = |a_m - L| + |L - a_n| \geq |(a_m - L) + (L - a_n)| = |a_m - a_n|$.
- This shows that the convergent sequence $<a_j>$ is Cauchy.

Note that the converse of this theorem also holds. That is, any sequence of real numbers that is Cauchy is a convergent sequence. This result will be proved in Sect. 3.7. An important and useful consequence of the above theorem is its contrapositive: *If a sequence is not Cauchy, then it does not converge.* Often when one wants to show that a sequence does not converge, one shows that there is some $\epsilon > 0$ such that for every N there are natural numbers m and n greater than N for which $|a_m - a_n| \geq \epsilon$.

Another important property of Cauchy sequences is that all Cauchy sequences are bounded. If the sequence $<a_n>$ is Cauchy, then there is a natural number N such that whenever $m, n \geq N$, the difference $|a_m - a_n| < 1$. The set $\{a_1, a_2, a_3, \ldots, a_N\}$

is a finite set, so it is bounded by some number, K. That is, $|a_n| \leq K$ for all $n \leq N$. If $m > N$, then, since both N and m are greater than or equal to N, it follows that $|a_m - a_N| < 1$ from which it follows that $|a_m| < |a_N| + 1 \leq K + 1$. Then the sequence $<a_n>$ is necessarily bounded above by $K+1$ and below by $-(K+1)$, and the sequence is bounded. A complete proof follows.

PROOF: All Cauchy sequences are bounded.

- Let $<a_n>$ be a Cauchy sequence.
- Then there is a natural number N such that for all $m, n \geq N$, $|a_m - a_n| < 1$.
- The set $\{a_1, a_2, a_3, \ldots, a_N\}$ is a finite set, so there is a K such that the set is bounded above by K and bounded below by $-K$.
- Let m be any natural number. If $m \leq N$, then $|a_m| \leq K$. If $m > N$, then $|a_m - a_N| < 1$, so $|a_m| = |a_m - a_N + a_N| \leq |a_m - a_N| + |a_N| < 1 + K$.
- It follows that all terms of the sequence lie between $-(K+1)$ and $K+1$, and, thus, the sequence is bounded.

One consequence of the last two results is that since all convergent sequences are Cauchy, all convergent sequences are bounded. The concept of a Cauchy sequence is not only applied to sequences of numbers but also to much more general sequences such as sequences of vectors, sequences of functions, and sequences of linear operators. Of course, one would need a way to discuss distances between the terms of a sequence in these other contexts, but when that makes sense, the concept of a Cauchy sequence becomes important.

3.5.8 Exercises

1. Which of the following sequences are monotone? Which of them are bounded above? Which of them are bounded below? Which of them are bounded?

   (a) $a_n = (-1)^n$
   (b) $a_n = \frac{n}{n+1}$
   (c) $a_n = 5^n$
   (d) $a_n = 5^{n^{(-1)^n}}$
   (e) $a_n = \frac{1+(-1)^n}{n+n^{-1}}$
   (f) $a_n = 5 - n(-1)^n$
   (g) $a_n = 1 \cdot \frac{1}{2} \cdot \frac{1}{3} \cdots \frac{1}{n}$

2. Write proofs of each of the following limits.

   (a) $\lim_{n\to\infty} \frac{6n}{3n+1} = 2$
   (b) $\lim_{n\to\infty} \frac{4n-1}{n+6} = 4$
   (c) $\lim_{n\to\infty} \frac{n^2+2n+1}{n^2-2n-5} = 1$