3. There is an integer k such that $f(x) \leq f(k)$ for all x between k and $k+1$.
4. For all $x > 0$ and all $y > 0$ there exists a $z < 0$ such that $f(z) \geq xf(y)$.

Prove that the following limits do not exist.

5. $f(x) = \frac{x}{|x|}$ as $x \to 0$
6. $f(x) = x \sin\left(\frac{1}{x-1}\right)$ as $x \to 1$
7. $f(x) = \left\{ \begin{array}{l} 5x \text{ if } x < 3 \\ 4x \text{ if } x \geq 3 \end{array} \right\}$ as $x \to 3$
8. $f(x) = \frac{4}{x^2-4}$ as $x \to 2$

3.7 Accumulation Points

A set A has an **accumulation point** p if for every $\epsilon > 0$ there is an $x \in A$ with $x \neq p$ and $|x - p| < \epsilon$. Informally, p is an accumulation point of A if there are points of A that are arbitrarily close to p. Note that the fact that p is an accumulation point of the set A has nothing to do with whether p is actually an element of A. For example, the set $A = \left\{\frac{1}{n} \mid n \in \mathbb{N}\right\}$ has one accumulation point, 0, because for every $\epsilon > 0$ there is an $n \in \mathbb{N}$ with $\frac{1}{n} < \epsilon$. Here the accumulation point 0 is not an element of the set A. The set $B = [0, 4]$ (the closed interval from 0 to 4) has infinitely many accumulation points. Indeed, every element of the interval B is an accumulation point of B because for each $x \in [0, 4]$ and each $\epsilon > 0$ there are infinitely many points in B within ϵ of x. Here all of the accumulation points of B are in B. Each point $x \in [0, 4]$ is also an accumulation point of the set $C = (0, 4) \cap \mathbb{Q}$, the set of rational numbers between 0 and 4. Here, some of the accumulation points are in C, and some are not. The set of natural numbers, $\mathbb{N}$, has no accumulation points. An element a of a set that is not an accumulation point of that set is called an **isolated point** of the set. For any isolated point a, there is an $\epsilon > 0$ such that a is the only element of the set in the interval $(a - \epsilon, a + \epsilon)$ (Fig. 3.9).

A word of warning is needed here. The term *accumulation point* is not used the same way by all authors. Many texts, especially those in Topology, will use the terms *limit point* or *cluster point* instead of *accumulation point*. Even more confusing is that some texts use the term *accumulation point* for something different.

Fig. 3.9 Set with accumulation point a and isolated point b

The first observation to make about accumulation points is that if p is an accumulation point of set A, then for every $\epsilon > 0$ there is not only one point of A within ϵ of p but infinitely many points of A within ϵ of p. The definition of accumulation point guarantees at least one point of A within ϵ of p, but once one point, $x \in A$, is found to be within ϵ of p, the definition also says that there must be another point $y \in A$ with $0 < |y - p| < |x - p|$. Since for each $x \in A$ close to p there must be another point $y \in A$ even closer to p, it follows that there are infinitely many points of A within ϵ of p.

Perhaps the most used fact about accumulation points is known as the Bolzano–Weierstrass Theorem which states that every infinite bounded set of real numbers has an accumulation point. As pointed out earlier, $\mathbb{N}$ has no accumulation points, and it is an infinite set. But $\mathbb{N}$ is not a bounded set. Intuitively, one cannot have a bounded infinite set without an accumulation point because one runs out of places to put the infinite number of points. If the points of a set are not allowed to bunch up anywhere, then one will not be able to find room for infinitely many of the points within a bounded interval.

There are several good strategies used to prove the Bolzano–Weierstrass Theorem, and two of those strategies are presented here. Of course, one only needs one good strategy to prove a theorem, but these proofs are instructive and use techniques commonly employed in Analysis proofs. One begins each proof with a statement about the set A being an infinite bounded set. Since A is a bounded set, it will have a lower bound, a, and an upper bound, b showing that $A \subseteq [a, b]$. The first strategy is to construct the set $S = \{x \geq a \mid [a, x] \cap A \text{ is finite}\}$, that is, a value $x \geq a$ is in the set S if there are finitely many element of A which fall in the interval $[a, x]$. First observe that the set S is an interval. This follows because if $y \in S$, then $[a, y] \cap A$ is finite, so if x is between a and y, then $[a, x] \cap A \subseteq [a, y] \cap A$ must also be finite, and $x \in A$. The next observation is that S is not empty because the point a, whether or not it is in A, is in S since $[a, a] \cap A$ contains at most one point, so it is finite. Since $[a, b] \cap A = A$ is an infinite set, the set S is bounded above by b. The Completeness Axiom now shows that S must have a least upper bound, p. It will follow that p is an accumulation point of A because for all $\epsilon > 0$, the set A will have only finitely many elements less than $p - \epsilon$ but infinitely many elements less than $p + \epsilon$ implying that there are infinitely many elements of A within ϵ of p. Here is the complete proof.

PROOF (Bolzano–Weierstrass Theorem): Every infinite bounded set of real numbers has an accumulation point.

- Let A be an infinite bounded set of real numbers.
- Because A is bounded, it has a lower bound, a, and an upper bound, b, showing that $A \subseteq [a, b]$.
- Define set $S = \{x \geq a \mid [a, x] \cap A \text{ is finite}\}$.
- Note that $a \in S$ since $[a, a] \cap A$ is finite, so S is nonempty.
- Note that if $z \geq b$, then $[a, z] \cap A = A$ is an infinite set, so $z \notin S$ showing that S is bounded above by b.
- By the Completeness Axiom, S has a least upper bound, p.
- Given $\epsilon > 0$, $p - \epsilon < p$ so $p - \epsilon$ is not an upper bound of S. Hence, there is a $y \in S$ with $y > p - \epsilon$. It follows that there are only finitely many elements of A less than or equal to y.
- Also, $p + \epsilon > p$, so $p + \epsilon \notin S$. It follows that $[a, p + \epsilon] \cap A$ is infinite.
- Thus, there must be infinitely many elements of A between $p - \epsilon$ and $p + \epsilon$, and there must be an element of A not equal to p within ϵ of p.
- This shows that p is an accumulation point of A.

The second strategy also begins with the interval $[a, b]$ that contains the infinite bounded set, A. One can rename the end points of this interval to be $a_1 = a$ and $b_1 = b$. Since $[a_1, b_1] \cap A = A$ is infinite, it follows that either $[a_1, \frac{a_1+b_1}{2}] \cap A$ or $[\frac{a_1+b_1}{2}, b_1] \cap A$ is an infinite set. If $[a_1, \frac{a_1+b_1}{2}] \cap A$ is infinite, define $a_2 = a_1$ and $b_2 = \frac{a_1+b_1}{2}$. Otherwise, define $a_2 = \frac{a_1+b_1}{2}$ and $b_2 = b_1$. In either case, $[a_2, b_2] \cap A$ is an infinite set. This procedure can be repeated so that for every $n \in \mathbb{N}$, one gets an interval $[a_n, b_n]$ where $[a_n, b_n] \cap A$ is infinite, and each interval is half the length of the previous interval. Also, the sequence of left endpoints, $<a_n>$, is a monotone increasing sequence bounded above by b, and the sequence of right endpoints, $<b_n>$, is a monotone decreasing sequence bounded below by a. Thus, both of these sequences converge. In fact, both of these sequences must converge to the same limit, p. This follows because the distances between the terms of the sequences, $b_n - a_n$, keep getting smaller and converge to 0. Given an $\epsilon > 0$, it will follow that there is an n such that a_n and b_n are both within ϵ of p. Thus, $(p - \epsilon, p + \epsilon) \cap A$ contains $[a_n, b_n] \cap A$ which is infinite. Here is the complete proof.

PROOF (Bolzano–Weierstrass Theorem): Every infinite bounded set of real numbers has an accumulation point.

- Let A be an infinite bounded set of real numbers.
- Because A is bounded, it has a lower bound, a_1, and an upper bound, b_1, showing that $A \subseteq [a_1, b_1]$ and $[a_1, b_1] \cap A$ is an infinite set.
- Define sequences $<a_n>$ and $<b_n>$ recursively as follows.
- Suppose, for natural number n, a_n and b_n have been defined so that $[a_n, b_n] \cap A$ is an infinite set. If $[a_n, \frac{a_n+b_n}{2}] \cap A$ is infinite, then define $a_{n+1} = a_n$ and $b_{n+1} = \frac{a_n+b_n}{2}$. Otherwise, define $a_{n+1} = \frac{a_n+b_n}{2}$ and $b_{n+1} = b_n$. In either case, $[a_{n+1}, b_{n+1}] \cap A$ is an infinite set.
- By the way the sequences are constructed, for each n it follows that $a_n \leq a_{n+1} < b_{n+1} \leq b_n$ showing that $<a_n>$ is a monotone increasing sequence bounded above by each b_i, and $<b_n>$ is a monotone decreasing sequence bounded below by each a_i.
- Also, by the way the sequences are constructed, for each n, $b_n - a_n = \frac{b_1-a_1}{2^{n-1}}$.
- Thus, the bounded monotone sequence $<a_n>$ must converge to a number p_a, and the bounded monotone sequence $<b_n>$ must converge to a number p_b. But $p_b - p_a \leq b_n - a_n = \frac{b_1-a_1}{2^{n-1}}$ and, therefore, $p_b - p_a$ must be zero. Let $p = p_a = p_b$, and note that for each n, $p \in [a_n, b_n]$.
- Given $\epsilon > 0$, select a natural number n such that $\frac{b_1-a_1}{2^{n-1}} < \epsilon$. Then $p - \epsilon < a_n \leq p \leq b_n < p + \epsilon$. Hence, $[a_n, b_n] \cap A \subseteq (p - \epsilon, p + \epsilon) \cap A$ is infinite showing that there is an element of A not equal to p but within ϵ of p.
- This shows that p is an accumulation point of A.

You now have the machinery necessary to prove the result mentioned in Sect. 3.6 that all Cauchy sequences converge. The difficulty in proving this result earlier was that given a Cauchy sequence $<a_n>$, it was not clear what real number would play the role of the limit L of the sequence. Now, the Bolzano–Weierstrass Theorem can provide an accumulation point to serve as this limit. There are two cases to consider. If the set of values in the sequence, $\{a_n\}$, is a finite set, then for the sequence to be Cauchy, the sequence will necessarily need to be constant from some point on, and, therefore, the sequence will converge. If the set of values in the sequence is infinite, then since all Cauchy sequences are bounded, the set of values in the sequence will be bounded and will have to have an accumulation point. It is then straightforward to show that the sequence converges to this accumulation point.