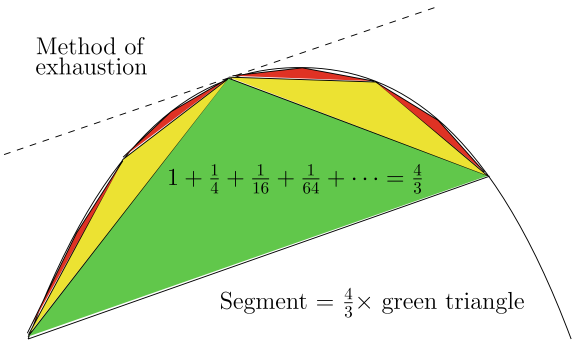

Fig. 6.1 Archimedes' parabolic segment

definition of area; that is the subject of measure theory. Nevertheless this will not stop us from discussing it, any more than it stopped the mathematicians of antiquity.

Archimedes calculated the area of a circle and the area of a parabolic segment (the figure bounded by a parabola and one of its chords). He gave the formula πr^2 for the circle and showed that $223/71 < \pi < 22/7$. His greatest achievement in the computation of area was the parabolic segment, stating that its area was $4/3$ times the area of a certain inscribed triangle (with base the given chord and top vertex at the point on the parabola where the tangent was parallel to the chord). To reach this conclusion he had to invent a method, the method of exhaustion, that in its use of an infinite sequence of approximations from below resembles modern integration theories. He also had to compute the sum of the geometric series $\sum_{n=0}^{\infty} 1/4^n$ (Fig. 6.1).

Fast forward to the fifteenth century and we find Kepler considering the volume of a wine barrel. This is a solid of revolution and the calculation of its volume depends on calculating the area of a plane figure.

Only with the invention of calculus was a method proposed that could be used to calculate the areas of general plane figures, starting with the area under the graph of a function. In the first place we consider the area between the graph of a positive function f and the x-axis, cut off by two vertical lines $x = a$ and $x = b$ (Fig. 6.2). This leads to the definition of the Riemann integral or the Darboux integral; two different approaches that turn out to be equivalent. We shall call it the Riemann–Darboux integral, although in defining it we shall take Darboux's approach.

We therefore proceed to Problem B and only later show how it leads to a solution to Problem A.

6.2 Defining the Riemann–Darboux Integral

Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Its domain is a bounded and closed interval. We do not assume that f is continuous. This is an advantage because it is necessary for practical applications to be able to integrate some discontinuous functions. But it is essential for the following considerations to make sense that f

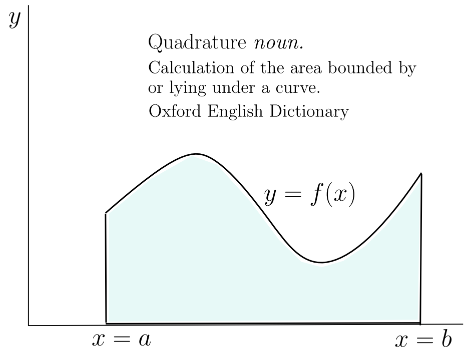

Fig. 6.2 Problem B. Calculate the area of a plane figure

should be bounded. This, and the requirement that the domain is a bounded and closed interval, are defects of the Riemann integral that were successfully removed by the introduction of the Lebesgue integral in the early twentieth century.

We do not assume that f is positive. However, in the case that $f(x) > 0$ for all x, the integral, when successfully defined, will give an acceptable notion for the area bounded by the lines $x = a$, $y = 0$, $x = b$ and the graph $y = f(x)$.

Definition A partition P of the interval $[a, b]$ is a finite sequence $(t_j)_{j=0}^m$ (not necessarily uniformly spaced), such that

$$a = t_0 < t_1 < t_2 < \cdots < t_m = b.$$

The intervals $[t_j, t_{j+1}]$ are called the subintervals of the partition.

For a given partition $(t_0, t_1, t_2, ..., t_m)$ we set

$$m_j = \inf_{[t_j, t_{j+1}]} f, \quad M_j = \sup_{[t_j, t_{j+1}]} f, \qquad j = 0, 1, ...m - 1$$

and define the lower sum $L(f, P)$ and the upper sum $U(f, P)$ by

$$L(f, P) = \sum_{j=0}^{m-1} m_j(t_{j+1} - t_j), \qquad U(f, P) = \sum_{j=0}^{m-1} M_j(t_{j+1} - t_j).$$

It is clear that $L(f, P) \le U(f, P)$, since $m_j \le M_j$ for each j.

Definition A partition P' is said to be finer than the partition P if every point of P is also a point of P'.

In the next three propositions we assume that f is a bounded function on the interval $[a, b]$.

Proposition 6.1 *Let P and P' be partitions of $[a, b]$. If P' is finer than P then*

$$L(f, P) \leq L(f, P') \leq U(f, P') \leq U(f, P).$$

Proof Consider how $L(f, P)$ changes if an additional point r is included in the partition. Suppose that $t_j < r < t_{j+1}$. The only change in $L(f, P)$ that arises is due to the replacement of the term $m_j(t_{j+1} - t_j)$ by the sum of two terms

$$m'_j(r - t_j) + m''_j(t_{j+1} - r),$$

where $m'_j = \inf_{[t_j, r]} f$ and $m''_j = \inf_{[r, t_{j+1}]} f$. But $m'_j \geq m_j$ and $m''_j \geq m_j$ (since the new infima are taken over smaller sets), so that

$$m'_j(r - t_j) + m''_j(t_{j+1} - r) \geq m_j(t_{j+1} - t_j),$$

and therefore $L(f, P) \leq L(f, P')$. The other inequality is proved by a similar argument. □

Proposition 6.2 *Let P_1 and P_2 be partitions of $[a, b]$. Then*

$$L(f, P_1) \leq U(f, P_2).$$

Proof Create a new partition P_3 by uniting the points in P_1 and P_2 into one sequence. Then P_3 is finer than P_1 and also finer than P_2. This implies that

$$L(f, P_1) \leq L(f, P_3) \leq U(f, P_3) \leq U(f, P_2),$$

so that $L(f, P_1) \leq U(f, P_2)$ as required. □

Consider next all numbers $L(f, P)$, that is, all lower sums, as P ranges over all possible partitions. These form a set (we could define it by specification for example). This set is moreover bounded above; for example, if we fix a partition P_1, then $L(f, P) \leq U(f, P_1)$ for every partition P. Similarly the set of all upper sums $U(f, P)$ is bounded below. We therefore define the lower and upper integrals

$$\underline{\int} f := \sup_P L(f, P), \qquad \overline{\int} f := \inf_P U(f, P)$$

as the supremum of the lower sums and the infimum of the upper sums respectively, taken over all possible partitions.

If f is a positive function and we wish to assign an area to the region between the graph $y = f(x)$ and the x-axis, bounded by the lines $x = a$ and $x = b$, then it seems clear that whatever this area might be, it should lie between the lower and upper integrals.

Proposition 6.3

$$\underline{\int} f \leq \overline{\int} f$$

Proof Let P_1 and P_2 be partitions of $[a, b]$. Then $L(f, P_1) \leq U(f, P_2)$. Taking the supremum over all partitions P_1, we obtain

$$\underline{\int} f \leq U(f, P_2).$$

Taking next the infimum over all partitions P_2, we obtain $\underline{\int} f \leq \overline{\int} f$ as required. □

Now we can define the Darboux integral. It has to be said that the process leading to this definition is remarkably short. As with the treatment of some previous concepts, such as limit or derivative, the definition singles out a class of functions, here called integrable, and for each integrable function defines a number called its integral.

Definition Let the function f be bounded on the interval $[a, b]$. If the upper and lower integrals of f are equal, we say that f is integrable (on the interval $[a, b]$). If f is integrable, the common value of its upper and lower integrals is called the integral of f (on the interval $[a, b]$). It is commonly denoted by one of the following:

$$\int f, \quad \int_{[a,b]} f, \quad \int_a^b f \quad \text{or} \quad \int_a^b f(x)\, dx.$$

6.2.1 Thoughts on the Definition

The concept of integral has a reputation for being hard to define. The definition we have just given for the Riemann–Darboux integral is actually quite short and some of its complexities may be concealed.

First of all the role of the completeness axiom comes out clearly in the repeated use of supremum and infimum. The supremum of the set of lower sums (defining the lower integral) is analogous to the supremum of a function. It is not though a function that assigns a real number to each real number in its domain, for the domain here is not a set of real numbers, but the set of partitions. The notation $L(f, P)$ reflects this and emphasises the dependence on P (whilst f remains fixed throughout the discussion).

It appears that the integral is essentially a more complex concept than the derivative. Previously the only sets we encountered were sets of real numbers, mainly intervals, or sets of natural numbers, and one could quite happily define the derivative without using more complex sets. When it comes to the integral, we have to embrace the set of all partitions of an interval. A partition is a sequence of real numbers with certain constraints; so the set of all partitions is a set of sequences of real numbers. This is a higher level of complexity than a set of real numbers. It seems that every approach to the integral involves complexity at this level.

Another approach to the integral is possible in which we approximate f from both above and below by step functions. We will then encounter *sets of step functions*.

Definition A function $g : [a, b] \to \mathbb{R}$ is called a step function if there exists a partition $(t_0, t_1, t_2, ..., t_m)$ of $[a, b]$, and numbers $(c_0, c_1, c_2, ..., c_{m-1})$, such that $g(x) = c_j$ for $t_j < x < t_{j+1}$, $j = 0, 1, 2, ..., m - 1$. In other words g is constant on each open interval $]t_j, t_{j+1}[$.

The area under the graph of a positive step function ought by rights to be $\sum_{j=0}^{m-1} c_j(t_{j+1} - t_j)$. This suggests that we first define the integral for the step function g, whether positive or not, as

$$S(g) = \sum_{j=0}^{m-1} c_j(t_{j+1} - t_j).$$

For a function f, supposed bounded on $[a, b]$, we can define the set of lower approximations as the set of all numbers $S(g)$ as g ranges through step functions such that $g \le f$ (it is here that a set of step functions is needed). This set is not empty thanks to the boundedness of f. Similarly the set of upper approximations is the set of all numbers $S(g)$ as g ranges through step functions such that $g \ge f$.

So far neither supremum nor infimum has been used. Next, we define the lower integral as the supremum of the set of all lower approximations and the upper integral as the infimum of the set of all upper approximations. Finally, the function is called integrable when the lower and upper integrals coincide.

The idea of approximating a function from above and below by simpler functions for which the integral has an obvious definition is common to many approaches to defining integrals. In particular it recurs in the definition of the Lebesgue integral, one of the greatest achievements of analysis in the twentieth century, to which the Riemann–Darboux integral is but a halfway house, and many of its faults are thereby alleviated.

Exercise Prove that the integral defined using approximation by step functions is the same as the Riemann–Darboux integral.

6.3 First Results on Integrability

The definition of the Riemann–Darboux integral raises some questions:

(a) What functions are integrable? More precisely, what conditions can we impose on f (in addition to its being bounded) that suffice for f to be integrable?
(b) Continuous functions on the interval $[a, b]$ are necessarily bounded. Are they integrable?

(c) Are step functions integrable? If so, and if f is a step function, is $\int f = S(f)$ (as defined in the last section)?
(d) If f is integrable can we find a practical way to calculate the integral? It is clearly impractical to compute the supremum over all lower sums.

We shall devote a considerable effort and a large part of this text to answering these questions.

One step is used repeatedly in the proofs and it is useful to set it out in advance. Let $\varepsilon > 0$. Since the lower integral is the supremum of the lower sums $L(f, P)$ over all partitions P, and the upper integral is the infimum of all upper sums $U(f, P)$ over all partitions P, there exists a partition P_1, such that

$$L(f, P_1) > \underline{\int} f - \varepsilon,$$

and another partition P_2, such that

$$U(f, P_2) < \overline{\int} f + \varepsilon.$$

Now construct a partition P by uniting the points of P_1 and P_2. Then P is simultaneously finer than both P_1 and P_2. Hence in passing from P_1 and P_2 to P, the lower sum cannot decrease and the upper sum cannot increase. Therefore the above inequalities hold also for P in place of P_1 and P_2.

The convenience is that both inequalities hold for the same partition. We can even do the same for a finite set of functions. For example, for two functions f and g, and a given ε, we can find a single partition P, such that the inequalities hold for both f and g.

6.3.1 Riemann's Condition

The condition introduced here is basic for proving that given functions are integrable.

Proposition 6.4 *The function f is integrable if and only if the following condition (which we shall call Riemann's condition[1]) is satisfied: for each $\varepsilon > 0$ there exists a partition P, such that*

$$U(f, P) - L(f, P) < \varepsilon.$$

Proof Assume that f is integrable. Then $\underline{\int} f = \overline{\int} f$. Choose a partition P, such that

[1] The name "Riemann's condition" appears in the book "Mathematical Analysis" by T. Apostol. I do not know of any other author who names it after Riemann. It is, however, convenient to have a name for it.

$$U(f, P) < \overline{\int} f + \frac{\varepsilon}{2} \quad \text{and} \quad L(f, P) > \underline{\int} f - \frac{\varepsilon}{2}.$$

It follows that

$$U(f, P) - L(f, P) < \overline{\int} f + \frac{\varepsilon}{2} - \underline{\int} f + \frac{\varepsilon}{2} = \varepsilon.$$

Conversely, assume that Riemann's condition is satisfied. Let $\varepsilon > 0$. Choose a partition P, such that $U(f, P) - L(f, P) < \varepsilon$. Now we have

$$L(f, P) \leq \underline{\int} f \leq \overline{\int} f \leq U(f, P)$$

so that $\overline{\int} f - \underline{\int} f < \varepsilon$. But this holds for all $\varepsilon > 0$. We conclude that $\overline{\int} f = \underline{\int} f$. □

The great strength of Riemann's condition is that we only have to find a single partition that satisfies $U(f, P) - L(f, P) < \varepsilon$. At this point it is useful to note that

$$U(f, P) - L(f, P) = \sum_{j=0}^{m-1} \Omega_j(f)(t_{j+1} - t_j)$$

where $\Omega_j(f)$ denotes the oscillation of f on the interval $[t_j, t_{j+1}]$, that is, the difference between the supremum and the infimum (see Sect. 4.5). We recall (Sect. 4.5 Exercise 4) that the oscillation of f on the interval $[c_1, c_2]$ is the same as the quantity

$$\sup_{c_1 \leq x, y \leq c_2} |f(x) - f(y)|.$$

The supremum here is taken over all pairs of points, x and y, in the interval $[c_1, c_2]$. This formula is very useful for comparing the oscillation of two functions, especially when it is required to deduce the integrability of one of them from the known integrability of the other, as we shall see.

6.3.2 *Integrability of Continuous Functions and Monotonic Functions*

We begin to answer the question as to which functions are integrable. We shall show that, loosely paraphrased, continuous functions and monotonic functions are integrable.

Proposition 6.5 *Let $f : [a, b] \to \mathbb{R}$ be continuous. Then f is integrable.*

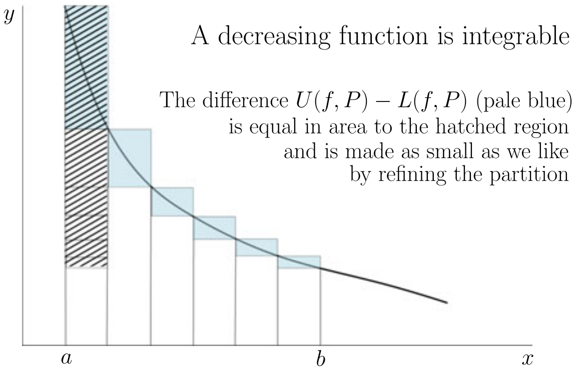

Fig. 6.3 Picture of the proof, adapted from Newton's Principia

Proof Let $\varepsilon > 0$. We use the small oscillation theorem (Proposition 4.14; now is the time to read it). There exists a partition P, such that $M_j - m_j < \varepsilon$ for each subinterval of the partition. But then

$$U(f, P) - L(f, P) = \sum_{j=0}^{m-1}(M_j - m_j)(t_{j+1} - t_j) < \sum_{j=0}^{m-1}\varepsilon(t_{j+1} - t_j) = \varepsilon(b - a)$$

and Riemann's condition is satisfied. □

Newton's pictorial proof of the integrability of monotonic functions is illustrated in Fig. 6.3.

Proposition 6.6 *Let $f : [a, b] \to \mathbb{R}$ be monotonic. Then f is integrable.*

Proof Assume for example that f is increasing (though not necessarily strictly). If $f(a) = f(b)$ then f is constant and obviously integrable; see the next section. So we may suppose that $f(a) < f(b)$.

Let $\varepsilon > 0$. Construct a partition $P = (t_0, t_1, ..., t_m)$, such that

$$t_{j+1} - t_j < \frac{\varepsilon}{f(b) - f(a)}$$

for $j = 0, 1, 2, ..., m$. Since f is increasing we have $m_j = f(t_j)$ and $M_j = f(t_{j+1})$, and we verify Riemann's condition by the calculation

$$\begin{aligned} U(f, P) - L(f, P) &= \sum_{j=0}^{m-1}(M_j - m_j)(t_{j+1} - t_j) \\ &= \sum_{j=0}^{m-1}(f(t_{j+1}) - f(t_j))(t_{j+1} - t_j) \\ &\leq \frac{\varepsilon}{f(b) - f(a)} \sum_{j=0}^{m-1}(f(t_{j+1}) - f(t_j)) \\ &\leq \frac{\varepsilon}{f(b) - f(a)}(f(b) - f(a)) = \varepsilon. \end{aligned}$$

□

6.3.3 Two Simple Integrals Computed

In this short section we shall compute our first integrals. The two results are not very impressive, and the treatment of the first function may seem tortuous, but a wait and see attitude is required. They will be used to find the integrals of step functions in Sect. 6.4.

Function A. Let $f : [a, b] \to \mathbb{R}$ where $f(x) = 0$ for $a < x < b$ but $f(a)$ and $f(b)$ are not necessarily 0. Then f is integrable and $\int f = 0$.

For each $\varepsilon > 0$ we consider the partition $P_\varepsilon = (a, a + \varepsilon, b - \varepsilon, b)$. If $f(a)$ and $f(b)$ are positive, then, for all ε, we have

$$U(f, P_\varepsilon) = \varepsilon(f(a) + f(b)).$$

If $f(a) > 0 \geq f(b)$, then, for all ε, we have

$$U(f, P_\varepsilon) = \varepsilon f(a).$$

If $f(b) > 0 \geq f(a)$, then, for all ε, we have

$$U(f, P_\varepsilon) = \varepsilon f(b).$$

Finally, if neither $f(a)$ nor $f(b)$ is positive, then, for all ε, we have

$$U(f, P_\varepsilon) = 0.$$

From these facts it is clear that

$$\overline{\int} f = \inf_P U(f, P) \leq \inf_{\varepsilon > 0} U(f, P_\varepsilon) = 0.$$

That is, $\overline{\int} f \leq 0$. Similar considerations apply to $L(f, P)$ and show that $\underline{\int} f \geq 0$. Hence $\underline{\int} f = \overline{\int} f = 0$ and so we have $\int f = 0$. The argument is illustrated in Fig. 6.4.

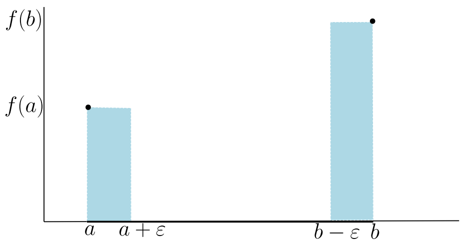

Fig. 6.4 An upper sum for function A

Function B. Let $g : [a, b] \to \mathbb{R}$ be the constant C. Then $\int g = C(b-a)$. Now $U(g, P) = L(g, P) = C(b-a)$ for every partition and therefore g is integrable with $\int g = C(b-a)$.

6.4 Basic Integration Rules

The rules proved in this section enable us to build new integrable functions from the old ones. Loosely described, the sum and product of integrable functions are integrable. Moreover integration is a linear operation in the space of functions integrable on a given interval.

In the preamble to rules and propositions, we shall often write that the functions are bounded before assuming that they are integrable. Though logically unnecessary, it could be useful to emphasise that Riemann–Darboux integration applies only to bounded functions.

Proposition 6.7 (Sum of functions) *Let $f : [a, b] \to \mathbb{R}$ and $g : [a, b] \to \mathbb{R}$ be bounded functions and assume that they are both integrable. Then $f + g$ is integrable and*

$$\int (f+g) = \int f + \int g.$$

Proof Let $P = (t_0, t_1, ..., t_m)$ be a partition of $[a, b]$. Set

$$m_j = \inf_{[t_j, t_{j+1}]} (f+g), \qquad m'_j = \inf_{[t_j, t_{j+1}]} f, \qquad m''_j = \inf_{[t_j, t_{j+1}]} g,$$

with similar definitions for M_j, M'_j, M''_j using suprema instead of infima.

For x in $[t_j, t_{j+1}]$ we have $f(x) + g(x) \le M'_j + M''_j$, so that we find $M_j \le M'_j + M''_j$. Similarly $m_j \ge m'_j + m''_j$. These give the inequalities

$$U(f+g, P) \le U(f, P) + U(g, P), \qquad L(f+g, P) \ge L(f, P) + L(g, P).$$

Let $\varepsilon > 0$. There exists a partition P (see the discussion in Sect. 6.3 on this point), such that

$$U(f, P) < \int f + \varepsilon, \qquad U(g, P) < \int g + \varepsilon$$

$$L(f, P) > \int f - \varepsilon, \qquad L(g, P) > \int g - \varepsilon.$$

We obtain

$$U(f+g, P) - L(f+g, P) \le U(f, P) - L(f, P) + U(g, P) - L(g, P) < 4\varepsilon.$$

This shows that Riemann's condition holds for $f + g$. In addition we have

$$\int f + \int g - 2\varepsilon < L(f, P) + L(g, P) \leq L(f + g, P)$$
$$\leq \int (f + g) \leq U(f + g, P) \leq U(f, P) + U(g, P) < \int f + \int g + 2\varepsilon$$

so that the inequality

$$\int f + \int g - 2\varepsilon < \int (f + g) < \int f + \int g + 2\varepsilon$$

holds for all $\varepsilon > 0$. We conclude that $\int (f + g) = \int f + \int g$. □

Proposition 6.8 (Multiplication by scalars) *Let $f : [a, b] \to \mathbb{R}$ be bounded and integrable. Let α be a real number. Then the function αf is integrable on $[a, b]$ and*

$$\int \alpha f = \alpha \int f.$$

Proof For an arbitrary set B we have the equalities

$$\sup_B(\alpha f) = \alpha \sup_B f, \quad \inf_B(\alpha f) = \alpha \inf_B f \quad (\alpha > 0) \tag{6.1}$$

and

$$\sup_B(\alpha f) = \alpha \inf_B f, \quad \inf_B(\alpha f) = \alpha \sup_B f \quad (\alpha < 0). \tag{6.2}$$

Hence

$$U(\alpha f, P) = \alpha U(f, P), \quad L(\alpha f, P) = \alpha L(f, P) \quad (\alpha > 0)$$

and

$$U(\alpha f, P) = \alpha L(f, P), \quad L(\alpha f, P) = \alpha U(f, P) \quad (\alpha < 0).$$

In the case $\alpha > 0$ we therefore have

$$\sup_P L(\alpha f, P) = \alpha \sup_P L(f, P) = \alpha \int f = \alpha \inf_P U(f, P) = \inf_P U(\alpha f, P).$$

The extreme terms are therefore equal. Hence each is the same as $\int \alpha f$ and at the same time $\alpha \int f$.

In the case $\alpha < 0$ we have

$$\sup_P L(\alpha f, P) = \alpha \inf_P U(f, P) = \alpha \int f = \alpha \sup_P L(f, P) = \inf_P U(\alpha f, P)$$

with the same conclusion. □

Exercise Prove the formulas (6.1) and (6.2) in the proof of Proposition 6.8.

Proposition 6.9 (Join of intervals) *Let* $f : [a, b] \to \mathbb{R}$ *be bounded and let* $a < c < b$*. If* f *is integrable on* $[a, c]$ *and also on* $[c, b]$ *then* f *is integrable on* $[a, b]$ *and*

$$\int_{[a,b]} f = \int_{[a,c]} f + \int_{[c,b]} f.$$

Conversely if f *is integrable on* $[a, b]$*, then* f *is also integrable on* $[a, c]$ *and on* $[c, b]$ *and the same equation holds.*

Proof Consider the first assertion. Let f be integrable both on $[a, c]$ and on $[c, b]$. Denote by f_1 the restriction of f to $[a, c]$ and by f_2 the restriction of f to $[c, b]$.

Let $\varepsilon > 0$. Choose partitions P_1 on $[a, c]$ and P_2 on $[c, b]$, such that

$$U(f_1, P_1) - \varepsilon < \int_{[a,c]} f < L(f_1, P_1) + \varepsilon$$

and

$$U(f_2, P_2) - \varepsilon < \int_{[c,b]} f < L(f_2, P_2) + \varepsilon.$$

Next construct a partition P on $[a, b]$ by uniting P_1 and P_2. It is clear that

$$L(f, P) = L(f_1, P_1) + L(f_2, P_2)$$

and

$$U(f, P) = U(f_1, P_1) + U(f_2, P_2).$$

But then we get

$$U(f, P) - 2\varepsilon < \int_{[a,c]} f + \int_{[c,b]} f < L(f, P) + 2\varepsilon.$$

This gives $U(f, P) - L(f, P) < 4\varepsilon$ and Riemann's condition is satisfied for f on $[a, b]$. This allows us to expand the last inequalities to

$$\int_{[a,c]} f + \int_{[c,b]} f - 2\varepsilon < L(f, P) \le \int_{[a,b]} f \le U(f, P) < \int_{[a,c]} f + \int_{[c,b]} f + 2\varepsilon$$

which are valid for all $\varepsilon > 0$. The first claim of the proposition now follows.

For the second assertion we must show that f_1 and f_2 are integrable given that f is integrable. Let $\varepsilon > 0$. We consider a partition P of $[a, b]$, which contains the point c and satisfies $U(f, P) - L(f, P) < \varepsilon$. From P we make in an obvious way partitions P_1 of $[a, c]$ and P_2 of $[c, b]$ which satisfy $U(f_1, P_1) - L(f_1, P_1) < \varepsilon$ and $U(f_2, P_2) - L(f_2, P_2) < \varepsilon$. □