Assignment 1.

General Relativity: Basics and Beyond (Excerpt)

(Chapter 5: Elementary Phenomenology, pp. 49–50)

Author: Robert M. Wald (and/or standard attribution as presented in the edition used)
Year of publication: 1984

Instruction

Please translate the following excerpt into English.

Excerpt

General relativity brought in a huge conceptual change regarding the nature of gravitation. It introduced a sophisticated model for possible space-times, required it to be dynamical and provided a specific equation determining space-times appropriate in various physical contexts. Within this model, the motion of test bodies under Newtonian gravitational force is understood as geodesics of corresponding space-time. This forms the basis for the solar system tests of general relativity. As we saw in the discussion of wave motion in geometrical optics approximation, light too responds to gravity following light-like geodesics. Apart from these test bodies implications, general relativity impacts compact stars and their stability, strongly suggests new types of objects called black holes, points to the possibility of a singular beginning for an expanding universe and makes a brand new prediction of gravitational waves. This chapter is arranged according to these different implications of the theory.

In the following, we use geometrized units: $c = 1$, $G = 1$, and the Einstein equation is taken in the form:

$$R_{\mu\nu} - (1/2) R g_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi T_{\mu\nu}$$

Geodesics and the Classic Tests

The first set of predictions were in the context of the solar system where Newtonian theory was applied and tested extensively. To make new predictions based on planetary motions being geodesics, we must first choose a space-time appropriate for our solar system. In Section 2.4 we have already introduced the idealized solar system. We noted that the appropriate space-time should be time-independent, spherically symmetric, and should satisfy the source-free Einstein equation in the region exterior to the Sun.

Since coordinates are arbitrary and have no direct physical interpretation, symmetry cannot be defined using specific coordinate transformations unless suitable coordinates are chosen. It is convenient to first consider infinitesimal symmetries.

Consider a vector field $\xi^\mu(x)$ which generates an infinitesimal coordinate transformation:

$x^\mu \to x'^\mu(x) := x^\mu + \varepsilon \xi^\mu(x)$

Under this transformation, the metric transforms as:

$$g'_{\mu\nu}(x + \varepsilon \xi) = (\partial x^\alpha/\partial x'^\mu)(\partial x^\beta/\partial x'^\nu)\, g_{\alpha\beta}(x)$$

$$\Rightarrow \delta g_{\mu\nu} := g'_{\mu\nu}(x) - g_{\mu\nu}(x) \approx -\varepsilon (\xi^\alpha \partial_\alpha g_{\mu\nu} + \partial_\mu \xi^\alpha g_{\alpha\nu} + \partial_\nu \xi^\alpha g_{\mu\alpha})$$

$$= -\varepsilon (\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu) =: -\varepsilon L_\xi g_{\mu\nu} \quad (5.2)$$

If $\delta g_{\mu\nu} = 0$ under this transformation, the vector field is called a Killing vector field and satisfies:

$$\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 0$$

The transformation is then an infinitesimal isometry. This means that moving along an infinitesimal curve in the direction of ξ does not change the metric.

It also implies that the metric is independent of the parameter s along the integral curve:

$$dx^\mu(s)/ds = \xi^\mu(x(s))$$

This is an ordinary differential equation and always has a local solution, so integral curves always exist for smooth vector fields.

However, it is not always possible to find a hypersurface Σ (an $(n-1)$-dimensional surface in an n-dimensional manifold) to which a given vector field is orthogonal. A vector field ξ^μ is hypersurface-orthogonal if:

$$0 = \xi_\lambda (\nabla_\mu \xi_\nu - \nabla_\nu \xi_\mu) + \text{cyclic permutations of } (\lambda\mu\nu)$$

This is a form of the Frobenius theorem.

We note that linear combinations of Killing vectors are also Killing vectors, and the commutator of two Killing vectors:

$$[\xi^\mu \partial_\mu , \eta^\nu \partial_\nu] = (\xi \cdot \nabla\eta^\alpha - \eta \cdot \nabla\xi^\alpha)\, \partial_\alpha$$

is again a Killing vector.

We are now ready to characterize static, spherically symmetric space-times.

A space-time is stationary if there exists a timelike Killing vector ξ. It is static if this vector field is hypersurface orthogonal. It is spherically symmetric if there exist three spacelike Killing vectors ξ_a such that:

$$[\xi_a, \xi_b] = \varepsilon^c{}_{ab}\, \xi_c$$

and the orbit of SO(3) through any point is a 2-sphere.

Let t be the parameter along the stationary Killing vector.

Assignment 2:

Intermediate Value Property (clean notes format)

(Jonathan Kane – Writing Proofs in Analysis, 2006, p. 129, Section 4.7.3)

Instruction

Please translate the following excerpt into Serbian,

Intermediate Value Property

Suppose a function f is defined on an interval containing c and d.

The graph of f passes through:

- (c, f(c))
- (d, f(d))

It may happen that the graph passes through every value y between f(c) and f(d) as x moves from c to d.

Example

Let:

f(x) = 2x^2 − 3

Then:

- f(1) = −1
- f(2) = 5

For any y between −1 and 5:

x = sqrt((y + 3)/2)

Then:

- x is between 1 and 2
- f(x) = y

Definition (Intermediate Value Property)

A function f defined on an interval [a, b] has the intermediate value property if:

For all c, d such that:

- a ≤ c ≤ d ≤ b

and for every y between:

- f(c) and f(d)

there exists x in [c, d] such that:

f(x) = y

Intermediate Value Theorem

If a function is continuous on [a, b], then it has the intermediate value property on that interval.

Intuition

If f is continuous on [a, b], meaning the graph can be drawn without lifting the pencil, then the function must pass through every intermediate y-value between f(c) and f(d).

Proof Idea

Let:

- f continuous on [a, b]
- c, d ∈ [a, b]
- y between f(c) and f(d)

Goal:
Find x ∈ [c, d] such that f(x) = y

Construction

Assume:
f(c) < y < f(d)

Define set:

$S = \{ x \in [c, d] \mid f(x) \leq y \}$

Then:

- $c \in S \rightarrow S$ is not empty
- d is an upper bound of S

So S has a least upper bound:

$s = \sup S$

Key argument

If $f(s) < y$:

- continuity ⇒ values near s still $< y$
- contradicts that s is an upper bound

If $f(s) > y$:

- continuity ⇒ values near s still $> y$
- contradicts that s is least upper bound

Therefore:

$f(s) = y$

So:
$x = s$

Other cases

- If $f(c) > f(d)$: same argument with reversed inequalities
- If $f(c) = f(d)$: trivial case, $x = c$

Conclusion

Without loss of generality, the proof assumes:

$f(c) < f(d)$

All other cases follow similarly.

Formal Source Note

This is an excerpt adapted from:

Jonathan Kane,
Writing Proofs in Analysis,
Springer, 2006,
Section 4.7.3: The Intermediate Value Property,
p. 129.

Assignment 3: GETTING TO KNOW YOU

(David Marusek – Getting to Know You, 2007, p. 114)

Instruction

Please translate the following excerpt into English.

Excerpt

Zoranna sighed. What do you need to know?
Shall Bug reprogram itself to enable Bug to process the file as requested?
No, Bug, I don't have the time to reprogram you, even if I knew how.
Shall Bug reprogram itself?
It could reprogram itself? Ted had failed to mention that feature. A tool they'd forgotten to disable? Yes, Bug, reprogram yourself.

A handicapped icon blinked on the cornice display, and the elevator's speed slowed to a crawl.

Thank you, Bug. That's more like it.

A jerry standing in the corner of the crowded elevator said, "Oh, lift?"
"Lift speed may not exceed five floors per minute," the elevator replied.

The jerry rose on tiptoes and surveyed his fellow passengers. "Right," he said, "who's the gimp?" Everyone looked at their neighbors. There were michelles, jennies, a pair of jeromes, and a half-dozen other germlines. They all looked at Zoranna, the only person not dressed in AP brown and teal.

"I'm sorry," she said, pressing her palm to her temple, "I have an aneurysm the size of a grapefruit. The slightest strain . . ." She winced theatrically.

"Then have it fixed!" the jerry said, to murmured agreement.

"Gladly," said Zoranna. "Could you pony me the OE23,000?"

The jerry har-harred and looked her up and down appraisingly. “Sweetheart, if you spent half as much money on the vitals as you obviously do on the peripherals,” he leered, “you wouldn’t have this problem, now would you?”

Zoranna had never liked the jerry type; they were spooky. In fact, more jerries had to be pithed in vatero for incipient sociopathy than any other commercial type. Professionally, they made superb grunts; most of the indentured men in the Protectorate’s commando forces were jerries. This one, however, wore an EXTRUSIONS UNLIMITED patch on his teal ball cap; he was security for a retail mall. “So,” he said, “where you heading?”

“Sub40?” she said.

Passengers consulted the cornice display and groaned. The jerry said, “At this rate it’ll take me an hour to get home.”

“Again I apologize,” said Zoranna, “but all the down lifts were spango. However, if everyone here consensed to drop me off first—?”

There was a general muttering as passengers spoke to their belts or tapped virtual keyboards, and the elevator said, “Consensus has been modified.”

Formal Source Note

This is an excerpt from:

David Marusek,
Getting to Know You,
2007,
p. 114.