One can proceed similarly with the curl operator: Calculating the circulation of the electric field $\boldsymbol{E}(\boldsymbol{r})$ along a *Stokes interface loop* (i.e., a small closed loop running in one direction on the upper side of the horizonal interface and in the opposite direction on the lower side but with negligible vertical height), one obtains from curl$\boldsymbol{E} = \nabla \times \boldsymbol{E} = 0$:

$$\boldsymbol{n} \times \left(\boldsymbol{E}^{(+)} - \boldsymbol{E}^{(-)} \right) = 0 \, . \tag{17.59}$$

From (17.58) and (17.59) one can derive a *law of refraction the electric field lines* at the interface between two different dielectric materials. This law follows from the fact that the tangential components of $\boldsymbol{E}$ are continuous, whereas the normal components

$$\boldsymbol{n} \cdot \boldsymbol{E}^{(i)} \quad \text{with} \quad i = 1, 2$$

(i.e., corresponding to the two different materials) are inversely proportional to the respective ε_i. It then follows that $\frac{\tan \alpha_2}{\tan \alpha_1} \equiv \frac{\varepsilon_2}{\varepsilon_1}$, with angles α_i to the normal. For $\varepsilon_2/\varepsilon_1 \to \infty$ one obtains conditions such as those for a metal surface *in vacuo*, $\alpha_2 \to 90°$, $\alpha_1 \to 0$ (a sketch is recommended).

18 Magnetic Field of Steady Electric Currents

18.1 Ampère's Law

For centuries it had been assumed that electricity and magnetism were completely separate phenomena. Therefore it was quite a scientific sensation when in 1818 the Danish physicist Hans Christian Ørsted proved experimentally that magnetic fields were not only generated by permanent magnetic dipoles, but also by electric currents, and when slightly later André Marie Ampère showed quantitatively that the *circulation of the magnetic field* $\boldsymbol{H}$ along a closed loop followed the simple relation:

$$\oint_{\partial F} \boldsymbol{H}(\boldsymbol{r}) \cdot \mathrm{d}\boldsymbol{r} = I(F) \quad \text{(Ampere's law)}\ . \tag{18.1}$$

Here, $I(F)$ is the *flux* of electric current through a surface F inserted into the closed loop $\Gamma = \partial F$[1]

$$I(F) := \iint_F \boldsymbol{j} \cdot \boldsymbol{n} \mathrm{d}^2 A\ . \tag{18.2}$$

$$\boldsymbol{j} := \varrho(\boldsymbol{r})\boldsymbol{v}(\boldsymbol{r})$$

is the vector of the *current density* (dimensionality: $\mathrm{A/cm^2} = \mathrm{C/(cm^2 s)}$).

With Stokes's integral theorem it follows that the differential form of Ampère's law (18.1) is given by:

$$\mathrm{curl}\boldsymbol{H} = \boldsymbol{j}\ . \tag{18.3}$$

For the special case of a thin wire aligned along the z-axis from $(-\infty)$ to $(+\infty)$, in which a steady electric current I flows, using cylindrical coordinates one obtains

$$\boldsymbol{H}_{\text{`z−wire'}} = \boldsymbol{e}_\varphi \frac{I}{2\pi r_\perp}\ . \tag{18.4}$$

Just as the electrostatic field of a point charge possesses a (three-dimensional) δ-divergence,

---

[1] The surface F is not uniquely defined by Γ, since different surfaces can be inserted into the same closed loop. This is the topological reason underlying *gauge freedom* of the vector potential, which is discussed below.

$$\operatorname{div}\left(\frac{q\boldsymbol{r}}{4\pi\varepsilon_0 r^3}\right) = q\delta(x,y,z) ,$$

an analogous relation is also valid for the curl of the magnetic field of the above "z-wire":

$$(\operatorname{curl}\boldsymbol{H}_{\text{“z−wire”}})(x,y,z) = I\delta(x,y)\boldsymbol{e}_z .$$

We can formulate these ideas in a general way:

The effective electric charges are the sources of the electrostatic field $\boldsymbol{E}(\boldsymbol{r})$ (whereas the vortices of $\boldsymbol{E}$ vanish); in contrast the vortices of the magnetostatic field $\boldsymbol{B}(\boldsymbol{r})$ correspond to effective electric currents (whereas the sources of $\boldsymbol{B}$ vanish).

Generally, a vector field $\boldsymbol{v}(\boldsymbol{r})$ is determined by its sources and vortices.

Note that we have written $\boldsymbol{B}$, not $\boldsymbol{H}$, and "effective" quantities, not "true" ones (see above). In particular, the relations between $\boldsymbol{E}$ and $\boldsymbol{D}$ as well as $\boldsymbol{B}$ and $\boldsymbol{H}$ are not quite simple, and not all magnetic fields are produced by electric currents (Sect. 18.5 → *spin magnetism*).

18.1.1 An Application: 2d Boundary Currents for Superconductors; The Meissner Effect

As already detailed in Sect. 17.2.9, at an interface Ampère's equation

$$\operatorname{curl}\boldsymbol{H} = \boldsymbol{j}$$

must be generalized to

$$\boldsymbol{n} \times (\boldsymbol{H}^+ - \boldsymbol{H}^-) = \boldsymbol{j}_s ,$$

where $\boldsymbol{j}_s$ is an *interface-current density* (dimensionality: A/cm, *not* A/cm^2; and we have $\boldsymbol{j}_s \equiv \sigma\boldsymbol{v}$, analogously to $\boldsymbol{j} \equiv \varrho\boldsymbol{v}$).

As we shall see, this formulation yields a simple explanation of the so-called Meissner effect of superconductivity. This effect amounts to "expelling" the magnetic field from the interior of a superconducting material, by loss-free interface (super)currents that flow tangentially at the interface between a superconducting region "1" (e.g., the r.h.s. of a plane) and a normally conducting region "2" (e.g., vacuum on the l.h.s.). For example, if the interface normal (from "1" to "2") is in the $(-x)$-direction and the external magnetic field (in the normal conducting region "2") is (as usual) in the $+z$-direction, then in "1" (at the interface towards "2") supercurrents flow in the y-direction, producing in "1" a field $-B\boldsymbol{e}_z$, which is different from zero only in a very thin layer of typical width

$$\Delta x = \lambda \approx 10\,\text{nm} .$$

For energy reasons (the magnetic field energy in region "1" can be saved) the supercurrents flow with such a strength that in the interior of region "1", outside the above-mentioned interface zone of width

$$\Delta x = \lambda ,$$

the external magnetic field is exactly compensated. Further details cannot be given here.

18.2 The Vector Potential; Gauge Transformations

Since

$$\mathrm{curl}\boldsymbol{H} = \boldsymbol{j}(\neq 0) ,$$

the magnetic field can no longer be calculated from a scalar potential: With

$$\boldsymbol{H}(\boldsymbol{r}) = -\mathrm{grad}\phi_m(\boldsymbol{r})$$

one would derive

$$\mathrm{curl}\boldsymbol{H} \equiv 0 ,$$

since

$$\mathrm{curl\ grad}\phi_m(\boldsymbol{r}) \equiv 0$$

for arbitrary scalar functions $\phi_m(\boldsymbol{r})$. ($\nabla \times (\nabla\phi_m)$ is formally a cross-product of two identical vectors and thus $\equiv 0$.) Fortunately we have

$$\mathrm{div}\boldsymbol{B}(\boldsymbol{r}) \equiv 0 ,$$

so that one can try:

$$\boldsymbol{B} = \mathrm{curl}\boldsymbol{A}(\boldsymbol{r}) ,$$

because

$$\mathrm{div\ curl}\boldsymbol{v}(\boldsymbol{r}) \equiv 0$$

for all vector fields $\boldsymbol{v}(\boldsymbol{r})$, as can easily be shown. (Formally div curl$\boldsymbol{v}$ is a so-called *spate product*, the determinant of a 3×3-matrix, i.e., of the form $\boldsymbol{u} \cdot [\boldsymbol{v} \times \boldsymbol{w}]$, with two identical vectors, $\nabla \cdot [\nabla \times \boldsymbol{v}]$, and therefore it also vanishes identically.)

In fact an important mathematical theorem, *Poincaré's lemma*, states the following: For source-free vector fields $\boldsymbol{B}$, i.e., if

$$\oiint_{\partial G} \mathrm{d}^2 A \boldsymbol{B} \cdot \boldsymbol{n} \equiv 0 ,$$

in a convex open region G (e.g., in the interior of a sphere) with a sufficiently well-behaved connected boundary ∂G, one can write vector potentials $\boldsymbol{A}$ with

$$\boldsymbol{B} = \mathrm{curl}\boldsymbol{A} .$$

One should note that $\boldsymbol{A}$ is not at all unique, i.e., there is an infinity of different vector potentials $\boldsymbol{A}$, but *essentially* they are all identical. If one adds an arbitrary gradient field to $\boldsymbol{A}$, then curl$\boldsymbol{A}$ is *not* changed at all. A so-called *gauge transformation*:

$$\boldsymbol{A} \to \boldsymbol{A}' := \boldsymbol{A} + \mathrm{grad} f(\boldsymbol{r}) \ , \tag{18.5}$$

with arbitrary $f(\boldsymbol{r})$, implies

$$\mathrm{curl}\boldsymbol{A} \equiv \mathrm{curl}\boldsymbol{A}' \ , \quad \text{since} \quad \mathrm{curl}\ \mathrm{grad} f \equiv 0 \ .$$

Therefore, the physical quantity $\boldsymbol{B}$ is unchanged.

18.3 The Biot-Savart Equation

In the following we consider, as usual, $G = \mathcal{R}^3$.

a) Firstly, we shall use a gauge such that div$\boldsymbol{A}(\boldsymbol{r}) = 0$ (*Landau gauge*).
b) Secondly, from Ampère's law,

$$\mathrm{curl}\boldsymbol{H} = \boldsymbol{j} \ , \quad \text{with} \quad \boldsymbol{B} = \mu_0 \boldsymbol{H} + \boldsymbol{J} \ ,$$

we conclude that

$$\mathrm{curl}\boldsymbol{B} = \mu_0 \boldsymbol{j} + \mathrm{curl}\boldsymbol{J} =: \mu_0 \boldsymbol{j}_B \ ,$$

with the effective current

$$\boldsymbol{j}_B := \boldsymbol{j} + \boldsymbol{M} \ , \quad \text{where} \quad \boldsymbol{M} := \frac{\boldsymbol{J}}{\mu_0}$$

is the *magnetization* and $\boldsymbol{J}$ the *magnetic polarization*[2].
c) Thirdly, we now use the general identity

$$\mathrm{curl}\ \mathrm{curl}\boldsymbol{A} \equiv \mathrm{grad}\ \mathrm{div}\boldsymbol{A} - \nabla^2 \boldsymbol{A} \ . \tag{18.6}$$

Hence, due to

$$\mathrm{curl}\ \boldsymbol{B} =: \mu_0 \boldsymbol{j}_B \ ,$$

the Cartesian components of $\boldsymbol{A}$ satisfy the Poisson equations

$$-\nabla^2 A_i = \mu_0 \cdot (j_B)_i \ , \quad \text{for} \quad i = x, y, z \ .$$

The solution of these equations is analogous to the electrostatic problem, *viz*

$$\boldsymbol{A}(\boldsymbol{r}) = \iiint \mathrm{d}V' \frac{\mu_0 \boldsymbol{j}_B(\boldsymbol{r}')}{4\pi |\boldsymbol{r} - \boldsymbol{r}'|} \ . \tag{18.7}$$

[2] In the cgs system the corresponding quantities are $\boldsymbol{M}' \left(= (\Delta V)^{-1} \sum_{\boldsymbol{r}_i \in \Delta V} \boldsymbol{m}_i' \right)$ and $4\pi \boldsymbol{M}'$.

One can easily show by partial integration that this result also satisfies the equation

$$\mathrm{div}\boldsymbol{A} \equiv 0\,, \quad \text{since} \quad \mathrm{div}\boldsymbol{j}_B = 0\,.$$

Later, in the context of the so-called *continuity equation*, this relation will be discussed more generally.

By applying the curl operator, equation (18.7) leads to the formula of *Biot and Savart*:

$$\boldsymbol{B}(\boldsymbol{r}) = \iiint \mathrm{d}V' \frac{\mu_0}{4\pi} \frac{\boldsymbol{j}_B(\boldsymbol{r}') \times (\boldsymbol{r}-\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|^3}\,. \tag{18.8}$$

In the integrand one has the same dependence on distance as in Coulomb's law for $\boldsymbol{E}$*, but complemented by the well-known right-hand rule connecting the directions of the effective current* $\boldsymbol{j}_B$ *and the magnetic induction* $\boldsymbol{B}$*, i.e., the product*

$$\frac{1}{\varepsilon_0}\varrho_E(\boldsymbol{r}')\frac{(\boldsymbol{r}-\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|^3}$$

is replaced by the cross-product

$$\mu_0 \boldsymbol{j}_B(\boldsymbol{r}') \times \frac{(\boldsymbol{r}-\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|^3}\,.$$

(It is no coincidence that the equation for $\boldsymbol{A}$, (18.7), is easier to remember than its consequence, the *Biot-Savart equation* (18.8).)

18.4 Ampère's Current Loops and their Equivalent Magnetic Dipoles

This section is especially important, since it shows that the relationships between electric currents and magnetic dipoles are very strong indeed. Firstly we state (without proof, but see the next footnote) that the magnetic induction $\boldsymbol{B}(\boldsymbol{r})$ produced by a *current loop* $\Gamma = \partial F$ (current I) is quantitatively identical to the magnetic field that would be produced by an infinitesimal film of magnetic dipoles inserted into the same loop, i.e., for the fictitious 2d-dipole density $\mathrm{d}\boldsymbol{m}$ of that film the following formula would apply:

$$\mathrm{d}\boldsymbol{m} \equiv \mu_0 I \boldsymbol{n} \mathrm{d}^2 A\,.$$

a) For a current loop, one obtains from *Biot and Savart's* equation

$$\boldsymbol{B}(\boldsymbol{r}) = \frac{\mu_0 I}{4\pi}\oint_{\partial F} \mathrm{d}\boldsymbol{r}' \times \frac{\boldsymbol{r}-\boldsymbol{r}'}{|\boldsymbol{r}-\boldsymbol{r}'|^3}\,. \tag{18.9}$$

b) In the dipole case one would obtain outside the (fictitious) dipole film the equivalent result

$$\boldsymbol{B} = \mu_0 \boldsymbol{H} \,,$$

with

$$\boldsymbol{H}(\boldsymbol{r}) = -\mathrm{grad}\frac{I}{4\pi}\iint_F \mathrm{d}^2A' \frac{\boldsymbol{n}(\boldsymbol{r}')\cdot(\boldsymbol{r}-\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|^3} \,. \qquad (18.10)$$

Proof of the equivalence of the two results proceeds analogously to Stokes's theorem, but since it is somewhat difficult in detail, we only give an outline in a footnote[3]. An example is given in Fig. 18.1.

In this context we additionally keep two useful identities in mind:

$$\boldsymbol{A} = \boldsymbol{m} \times \frac{\boldsymbol{r}}{4\pi r^3}$$

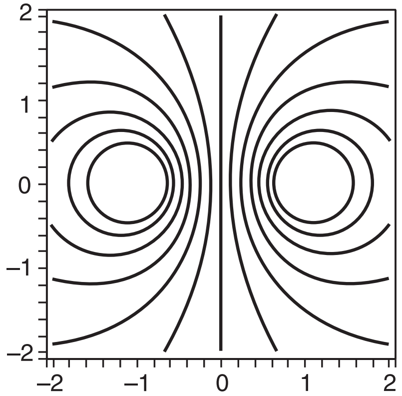

Fig. 18.1. The diagram illustrates a typical section of the magnetic field lines produced by a current loop of two (infinitely) long straight wires. The wires intersect the diagram at the points $(\pm 1, 0)$. The plane of the loop of area A $(\to \infty)$ and carrying a current I (i.e., of opposite signs in the two long wires) is perpendicular to the plane of the diagram. Exactly the same induction $\boldsymbol{B}(=\mu_0\boldsymbol{H})$ is also produced by a layer of magnetic dipoles inserted into the current loop, with the quantitative relation, $\mathrm{d}\boldsymbol{m} \equiv \mu_0 I \boldsymbol{n}\mathrm{d}^2A$, given in the text

[3] In the following we use the antisymmetric unit-tensor e_{ijk} and Einstein's summing convention, i.e., all indices which appear twofold are summed over. With these conventions Stokes's theorem becomes: $\oint_{\partial F} E_j \mathrm{d}x_j = \iint_F e_{jlm}\partial_l E_m n_j \mathrm{d}^2A$. Now the following chain of equations is true: $\oint_{\partial F} e_{ijk}\frac{\mathrm{d}x'_j(x_k-x'_k)}{|\boldsymbol{r}-\boldsymbol{r}'|^3}\left(\equiv \oint_{\partial F}\mathrm{d}x'_j e_{jki}\partial'_k\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}\right) = \iint_F e_{jlm}\partial'_l e_{mki}\partial'_k\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}n_j\mathrm{d}^2A' = -\iint_F e_{ikm}e_{jlm}\partial'_{lk}\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}n_j\mathrm{d}^2A'$. With the basic identity $e_{ikm}e_{jlm} = \delta_{ij}\delta_{kl} - \delta_{il}\delta_{kj}$ and the simple relations $\partial'_i\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|} = -\partial_i\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}$ and $\partial_{kk}\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|} = 0$ (for $\boldsymbol{r}' \neq \boldsymbol{r}$) our statement of equivalence is obtained.

for the vector potential of a magnetic dipole and

$$\text{curl}\left(\boldsymbol{m}\times\frac{\boldsymbol{r}}{r^3}\right)=\boldsymbol{m}\text{div}\frac{\boldsymbol{r}}{r^3}-\text{grad}\frac{\boldsymbol{m}\cdot\boldsymbol{r}}{r^3}$$

(cf. problem 5 of the exercises, summer 2002 [2]).

Because of the above-mentioned equivalence it would be natural to suggest that *all* magnetic dipole moments are generated in this way by Ampèrian current loops. However, this suggestion would be wrong: *There are magnetic moments which cannot be generated in this "classical" way, but which are related to the non-classical concept of "electron spin" (see Part III: Quantum Mechanics).* The following section deals with the difference.

18.5 Gyromagnetic Ratio and Spin Magnetism

An atomic electron orbiting the nucleus on a circular path of radius R with velocity

$$v=\omega R \quad \left(\text{time period} \quad T=\frac{2\pi}{\omega}\right)$$

has an angular momentum of magnitude

$$L=R\cdot m_{\text{e}}v=m_{\text{e}}\omega R^2\ .$$

According to the Ampèrian "current loop" picture it would be equivalent to a magnetic dipole moment

$$m=\mu_0\frac{e}{T}\pi R^2=\frac{\mu_0 e\omega R^2}{2}\ ,$$

where we have used $I=\frac{e}{T}$ (m_{e} is the electron mass).

For a current loop, therefore, the gyromagnetic ratio

$$\gamma:=\frac{m}{L}$$

is given by

$$\gamma\equiv\frac{\mu_0 e}{2m_{\text{e}}}\ .$$

However, in the nineteen-twenties due to an experiment by *Einstein and de Haas* it was shown that for the usual magnetic materials, e.g., alloys of Fe, Co and Ni, the gyromagnetic ratio is *twice* as large as the above ratio. For these materials the magnetism is due almost entirely to pure *spin magnetism*. For the angular momentum of these alloys the "classical" orbital contribution (see Part I) is almost negligible; the (dominant!) contribution is essentially "non-classical", i.e., due to *spin magnetism*, which is only understandable in a quantum mechanical context. (In fact, a profound analysis is not even possible in non-relativistic quantum mechanics, but only in Dirac's relativistic version.)

In elementary texts one often reads that the spin angular momentum of a (charged) particle is some kind of "proper angular momentum", this being acceptable if one does not consider the particle to be rotating like a "spinning top", since a spinning charge would have the classical value,

$$\gamma = \frac{\mu_0}{2m_e} ,$$

for the gyromagnetic ratio and *not* twice this value. One has to admit that these relations are complicated and not understandable at an elementary level.

19 Maxwell's Equations I: Faraday's Law of Induction; the Continuity Equation; Maxwell's Displacement Current

Maxwell's first and second equations, $\mathrm{div}\boldsymbol{D} = \varrho$ and $\mathrm{div}\boldsymbol{B} = 0$ (i.e., Gauss's law for the electric and the (non-existent) magnetic charges, respectively) also apply without change for time-dependent electrodynamic fields. This is different with respect to the third and fourth Maxwell equations:

a) *Faraday's law of induction* (Faraday 1832)

$$\mathrm{curl}\boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \,, \tag{19.1}$$

and

b) *Ampère's law including Maxwell's displacement current*:

$$\mathrm{curl}\boldsymbol{H} = \boldsymbol{j} + \frac{\partial \boldsymbol{D}}{\partial t} \,. \tag{19.2}$$

These two equations, (19.1) and (19.2), will be discussed in the following subsections. To aid our understanding of the last term in (19.2), known as Maxwell's displacement current, we shall include a subsection on the *continuity equation*. This general equation contains an important conservation law within it, the conservation of total charge (see below).

19.1 Faraday's Law of Induction and the Lorentz Force; Generator Voltage

In 1832 Faraday observed that a time-dependent change of magnetic flux

$$\phi_B(F) = \iint_F \boldsymbol{B} \cdot \boldsymbol{n} \mathrm{d}^2 A$$

through a current loop $\varGamma = \partial F$ gives rise to an *electromotive force* (i.e., a force by which electric charges of different sign are separated). This corresponds to a *generator voltage* which is similar to the off-load voltage between the two poles of a battery. (In the interior of a battery the current flows from the minus pole to the plus pole; only subsequently, in the external load circuit, does the current flow from plus to minus.) In Fig. 19.1 we present a sketch of the situation.

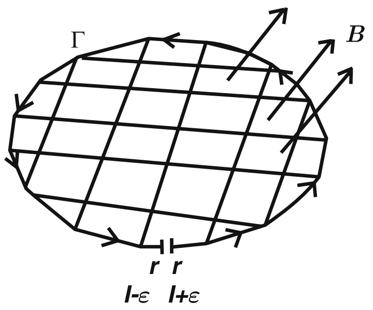

Fig. 19.1. Sketch to illustrate Faraday's law of induction. The three arrows on the r.h.s. of the figure denote a magnetic induction $\boldsymbol{B}$. An oriented loop Γ is plotted, as well as a paved surface F, which is inserted into Γ ($\Gamma = \partial F$) but which does not need to be planar as in the diagram. A change in magnetic flux $\phi_B(F) := \iint_F \boldsymbol{B}\cdot\boldsymbol{n}\mathrm{d}^2A$ gives rise to an induced voltage $U_i(\Gamma)\left(=\oint_{\boldsymbol{r}_{1+\varepsilon}}^{\boldsymbol{r}_{1-\varepsilon}} \boldsymbol{E}\cdot\mathrm{d}\boldsymbol{r}\right)$ between two infinitesimally close points $\boldsymbol{r}_{1+\varepsilon}$ and $\boldsymbol{r}_{1-\varepsilon}$ on the loop. These two points – which are formally the initial and end points of the loop – can serve as the poles of a voltage generator (the initial point $\boldsymbol{r}_{1+\varepsilon}$ corresponds to the negative pole). The related quantitative equation is Faraday's law: $U_i(t) = -\frac{\mathrm{d}\phi_B(F)}{\mathrm{d}t}$

Faraday's law of induction states that the *induced voltage* (i.e., the *generator voltage* mentioned above)

$$U_i = \oint_{\boldsymbol{r}_{1+\varepsilon}}^{\boldsymbol{r}_{1-\varepsilon}} \boldsymbol{E}(\boldsymbol{r})\cdot\mathrm{d}\boldsymbol{r}$$

between (arbitrary) initial points $\boldsymbol{r}_{1+\varepsilon}$ and (almost) identical end points $\boldsymbol{r}_{1-\varepsilon}$ of an (almost) closed line[1] $\Gamma = \partial F$ obeys the following law:

$$U_i(t) = -\frac{\mathrm{d}\phi_B(F)}{\mathrm{d}t}\,. \tag{19.3}$$

As already mentioned, the initial and end points of the (almost) closed loop Γ differ only infinitesimally. They correspond to the minus and plus poles of the generator, i.e., U_i is the generator voltage.

It does not matter at which position of the curve Γ the voltage is "tapped", nor does it matter whether the change of the magnetic flux results

a) from a change of Γ (i.e., form or size) *relative* to the measuring equipment,
b) from a change of the magnetic induction $\boldsymbol{B}(\boldsymbol{r},t)$, or
c) by a combination of both effects.

[1] A sketch is recommended. The normal vector $\boldsymbol{n}$ of the area F should coincide with the orientation of the loop $\Gamma = \partial F$.