

УАТ-решения задания за вѣсѣ

1. (C_*, d_*) je lanchaniti kompleks $\Rightarrow d_{n-1} \circ d_n = 0, \forall n \in \mathbb{Z}$

In je geef. w: $\partial_n(f)(x) = d_n(f(x))$,

wj. $\mathcal{I}_n(f) = d_n \circ f$ (*)

Kotremo ga pokazujemo ga je $\partial_{n-1} \circ \partial_n = 0$.

Нека је $f \in D_n = \text{Hom}(A, C_n)$.

$$(\partial_{n-1} \circ \partial_n)(f) = \partial_{n-1}(\partial_n(f)) \stackrel{(*)}{=} \partial_{n-1}(d_n \circ f) \stackrel{(*)}{=} \underbrace{d_{n-1} \circ d_n}_{=0} \circ f = 0$$

$$\Rightarrow \partial_{n-1} \circ \partial_n = 0 \quad \square$$

2. $C_*: \dots \rightarrow \mathbb{Z}_6 \xrightarrow{d_3} \mathbb{Z}_6 \xrightarrow{d_2} \mathbb{Z}_6 \xrightarrow{d_1} \mathbb{Z}_6 \xrightarrow{d_0} \mathbb{Z}_6 \rightarrow \dots$
 $\quad \quad \quad \parallel \quad \quad \quad \parallel \quad \quad \quad \parallel \quad \quad \quad \parallel \quad \quad \quad \parallel$
 $\quad \quad \quad C_3 \quad \quad \quad C_2 \quad \quad \quad C_1 \quad \quad \quad C_0 \quad \quad \quad C_{-1}$

саму право дакле кретеио индексирати

убери что осталось: $C_n = \mathbb{Z}_6, n \in \mathbb{Z}$

$$d_n = \begin{cases} \cdot 2, & n \text{ нечетно} \\ \cdot 3, & n \text{ четно} \end{cases}$$

$$H_n(C_*) \stackrel{\text{def}}{=} \ker d_n / \text{im } d_{n+1}$$

1° n натур $d_n = -2, \quad d_{n+1} = -3$

$$x \in \ker d\eta \Leftrightarrow d\eta(x) = 0 \Leftrightarrow 2x = 0 \Leftrightarrow x \in \{0, 3\}$$

$$\begin{array}{c} \uparrow \\ y \mathbb{Z}_6 \end{array}$$

↑
sep

$$2 \cdot 0 = 0 \quad \checkmark$$

$$2 \cdot 3 = 6 \equiv_6 0 \quad \checkmark$$

$$\Rightarrow \ker d_n = \{0, 3\} \cong \mathbb{Z}_2$$

$\text{im } d_{n+1} = ?$ Требамо сваке ел. из $\mathbb{Z}_6$ у нису се сврхе:

$$\left. \begin{array}{l} 0 \mapsto 0 \\ 1 \mapsto 3 \\ 2 \mapsto 6 \equiv_6 0 \\ 3 \mapsto 9 \equiv_6 3 \\ 4 \mapsto 12 \equiv_6 0 \\ 5 \mapsto 15 \equiv_6 3 \end{array} \right\} \Rightarrow \text{im } d_{n+1} = \{0, 3\} \cong \mathbb{Z}_2$$

$$\Rightarrow H_n(C_*) \cong \ker d_n / \text{im } d_{n+1} = \mathbb{Z}_2 / \mathbb{Z}_2 = 0$$

2° n неједнако $d_n = \cdot 3, d_{n+1} = \cdot 2$

$$x \in \ker d_n \Leftrightarrow d_n(x) = 0 \Leftrightarrow 3x = 0 \Leftrightarrow x \in \{0, 2, 4\}$$

$$\uparrow \\ 3\mathbb{Z}_6$$

$$\uparrow$$

$$3 \cdot 0 = 0 \quad \checkmark$$

$$3 \cdot 2 = 6 \equiv_6 0 \quad \checkmark$$

$$3 \cdot 4 = 12 \equiv_6 0 \quad \checkmark$$

$$\Rightarrow \ker d_n = \{0, 2, 4\} \cong \mathbb{Z}_3$$

$$\text{im } d_{n+1} = ?$$

$$\left. \begin{array}{l} 0 \mapsto 0 \\ 1 \mapsto 2 \\ 2 \mapsto 4 \\ 3 \mapsto 6 \equiv_6 0 \\ 4 \mapsto 8 \equiv_6 2 \\ 5 \mapsto 10 \equiv_6 4 \end{array} \right\} \Rightarrow \text{im } d_{n+1} = \{0, 2, 4\} \cong \mathbb{Z}_3$$

$$\Rightarrow H_n(C_*) \cong \ker d_n / \text{im } d_{n+1} = \mathbb{Z}_3 / \mathbb{Z}_3 = 0$$

Коначно, $H_n(C_*) = 0$, за све $n \in \mathbb{Z}$. $\square$

$$3. \quad C_* : \quad \cdots \rightarrow \mathbb{Z}_5 \xrightarrow{\substack{\cdot 2 \\ d_2}} \mathbb{Z}_5 \xrightarrow{\substack{0 \\ d_1}} \mathbb{Z}_5 \xrightarrow{\substack{\cdot 2 \\ d_0}} \mathbb{Z}_5 \xrightarrow{0} \cdots$$

$$\quad \quad \quad \parallel \quad \quad \parallel \quad \quad \parallel \quad \quad \parallel$$

$$\quad \quad \quad C_2 \quad \quad C_1 \quad \quad C_0 \quad \quad C_{-1}$$

Замеч, $C_n = \mathbb{Z}_5$, $d_n = \begin{cases} \cdot 2, & n \text{ парно} \\ 0, & n \text{ не парно} \end{cases}$

1° n парно $d_n = \cdot 2$, $d_{n+1} = 0$

как у зап. 2 ... $\ker d_n = \{0\} = 0$

↑
оснастка за
привуцаааааа
пруууу

$\text{im } d_{n+1} = 0$

$\Rightarrow H_n(C_*) \cong \ker d_n / \text{im } d_{n+1} = 0/0 = 0$

2° n не парно $d_n = 0$, $d_{n+1} = \cdot 2$

$\ker d_n = \mathbb{Z}_5$

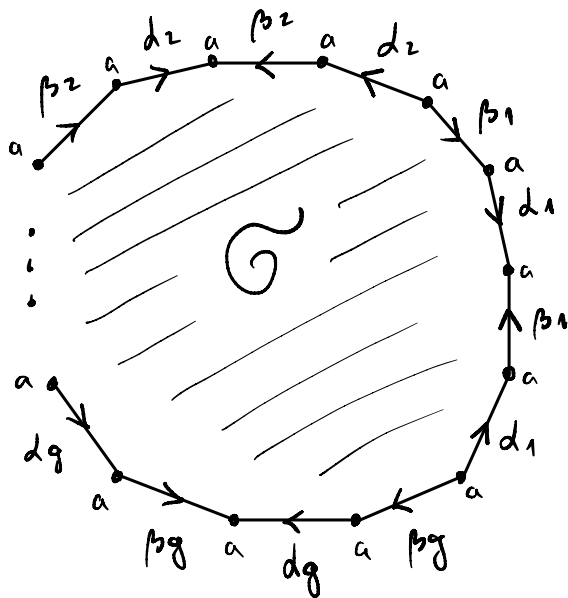
$\text{im } d_{n+1} = ?$

$$\left. \begin{array}{l} 0 \xrightarrow{\cdot 2} 0 \\ 1 \xrightarrow{\cdot 2} 2 \\ 2 \xrightarrow{\cdot 2} 4 \\ 3 \xrightarrow{\cdot 2} 6 \equiv_5 1 \\ 4 \xrightarrow{\cdot 2} 8 \equiv_5 3 \end{array} \right\} \Rightarrow \text{im } d_{n+1} = \{0, 1, 2, 3, 4\} \cong \mathbb{Z}_5$$

$\Rightarrow H_n(C_*) \cong \ker d_n / \text{im } d_{n+1} = \mathbb{Z}_5 / \mathbb{Z}_5 = 0$

Конец, $H_n(C_*) = 0$, за все $n \in \mathbb{Z}$. $\square$

4. M_g OT imamo ravninski model za M_g :



Ovo je u stvari kenijski kompleks.

0-kenije: a

1-kenije: $d_1, \beta_1, d_2, \beta_2, \dots, d_g, \beta_g$

2-kenije: σ

Protivory M_g prikupljajemo laticu kompleksa:

$$C_*: \dots \rightarrow 0 \xrightarrow[d_0]{d_1} \mathbb{Z} \xrightarrow[d_2]{d_1} \mathbb{Z}^{2g} \xrightarrow[d_3]{d_1} \mathbb{Z} \xrightarrow[d_4]{d_0} 0$$

$\underbrace{\quad}_{\text{dve cy 0 obge}} \quad \parallel \quad \parallel \quad \parallel$
 $C_2 \quad C_1 \quad C_0$

Preuzimanje:

$$C_*: 0 \rightarrow \mathbb{Z}\langle \sigma \rangle \xrightarrow{d_2} \mathbb{Z}\langle d_1 \rangle \oplus \mathbb{Z}\langle \beta_1 \rangle \oplus \dots \oplus \mathbb{Z}\langle d_g \rangle \oplus \mathbb{Z}\langle \beta_g \rangle \xrightarrow{d_1} \mathbb{Z}\langle a \rangle \rightarrow 0$$

Samo smo imali generatore izmjenjivog $\uparrow$

$$H_n(M_g) \stackrel{\text{def}}{=} H_n(C_*) = \ker d_n / \text{im } d_{n+1}$$

Kako je $C_n = 0$ za $n \neq 0, 1, 2$, tako je $\ker d_n = 0$ za $n \neq 0, 1, 2$ (jer $\ker d_n \leq C_n$), pa je $H_n(M_g) = 0$ za $n \neq 0, 1, 2$.
 Tada ga određujemo H_0, H_1 i H_2 .

$n=0$ M_g je puno povezan $\Rightarrow H_0(M_g) \cong \mathbb{Z}$

$$n=1 \quad \text{ker } d_1 = ?$$

$$d_1(\alpha_1) = d_1(\beta_1) = \dots = d_1(\alpha_g) = d_1(\beta_g) = \underbrace{a - a} = 0$$

дн слика све генераторе у 0 $\Rightarrow$ $ch \equiv 0$ ^{крај може почитати}

$$\Rightarrow \ker d_1 = C_1 = \mathbb{Z}^{2g}$$

$$\text{im } d_2 = ?$$

$$d_2(\sigma) = d_1 + \beta_1 - d_1 - \beta_1 + \dots + d_g + \beta_g - d_g - \beta_g = 0$$

d_2 — число перестановок σ у $0 \Rightarrow d_2 \equiv 0 \Rightarrow \text{im } d_2 = 0$

$$\Rightarrow H_1(M_g) \cong \ker d_1 / \operatorname{im} d_2 = \mathbb{Z}^{2g} / 0 = \mathbb{Z}^{2g}$$

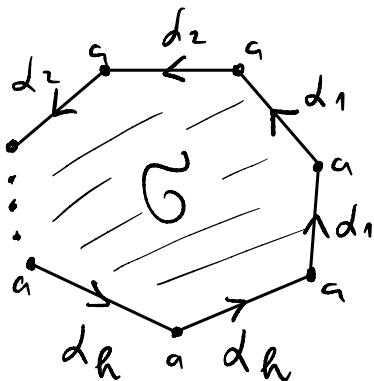
$$n=2 \quad d_2=0 \Rightarrow \ker d_2 = C_2 = \mathbb{Z}$$

$$\text{im } d_3 = 0$$

$$\Rightarrow H_2(M_g) \cong \ker d_2 / \operatorname{im} d_3 = \mathbb{Z}/0 = \mathbb{Z}.$$

Конечно, $H_n(M_g) \cong \begin{cases} \mathbb{Z}, & n=0, 2 \\ \mathbb{Z}^{2g}, & n=1 \\ 0, & \text{иначе} \end{cases}$

5. liegen 3G N_h μ OT:



○-kerije : a

1-kesye: $d_1, d_2, \dots, d_n$

2-kerije : 2

$$C_*: \quad 0 \rightarrow \mathbb{Z} \xrightarrow{d_2} \mathbb{Z}^h \xrightarrow{d_1} \mathbb{Z} \rightarrow 0$$

$$\quad \quad \quad \downarrow \quad \quad \quad \downarrow \quad \quad \quad \downarrow$$

$$\quad \quad \quad C_2 \quad \quad \quad C_1 \quad \quad \quad C_0$$

нј: $C_*: 0 \rightarrow \mathbb{Z}\langle \sigma \rangle \xrightarrow{d_2} \mathbb{Z}\langle d_1 \rangle \oplus \dots \oplus \mathbb{Z}\langle d_h \rangle \xrightarrow{d_1} \mathbb{Z}\langle a \rangle \rightarrow 0$

$$C_n = 0, n \neq 0, 1, 2 \Rightarrow H_n(N_n) = 0, n \neq 0, 1, 2$$

$n=0$ N_n је нулта група $\Rightarrow H_0(N_n) \cong \mathbb{Z}$

$n=1$ $d_1(d_1) = \dots = d_1(d_h) = a - a = 0 \Rightarrow d_1 = 0 \Rightarrow \ker d_1 = C_1 = \mathbb{Z}^h$

$$d_2(\sigma) = d_1 + d_1 + \dots + d_h + d_h = 2(d_1 + \dots + d_h)$$

Елементи у $C_2 = \mathbb{Z}\langle \sigma \rangle$ су облика $k\sigma$, $k \in \mathbb{Z}$, то је

$$\text{im } d_2 = \{ d_2(k\sigma) \mid k \in \mathbb{Z} \} = \{ k \cdot 2(d_1 + \dots + d_h) \mid k \in \mathbb{Z} \} =$$

$$= \mathbb{Z}\langle 2(d_1 + \dots + d_h) \rangle \quad (\text{јер, } \text{im } d_2 \cong \mathbb{Z}, \text{ а овде имамо}$$

$$\text{мо само један генератор}$$

$$2(d_1 + \dots + d_h) \text{ ове групе})$$

Сада користићемо представљање групе преко генератора и релација (подељити се из дефиниције за ОТ стр. 69-73)

$$H_1(N_n) \cong \ker d_1 / \text{im } d_2 \cong \mathbb{Z}\langle d_1 \rangle \oplus \dots \oplus \mathbb{Z}\langle d_h \rangle / \mathbb{Z}\langle 2(d_1 + \dots + d_h) \rangle$$

$$\cong \langle d_1 | - \rangle \oplus \dots \oplus \langle d_h | - \rangle / \mathbb{Z}\langle 2(d_1 + \dots + d_h) \rangle \cong$$

$$\cong \text{Ab} \langle d_1, \dots, d_h | - \rangle / \mathbb{Z}\langle 2(d_1 + \dots + d_h) \rangle \cong$$

$$\begin{aligned}
&\cong A\mathbb{G} \langle d_1, \dots, d_n \mid \underbrace{2(d_1 + \dots + d_n)}_{\beta} = 0 \rangle \cong \\
&\cong A\mathbb{G} \langle d_1, \dots, d_n, \beta \mid 2\beta = 0, \beta = d_1 + \dots + d_n \rangle \cong \\
&\cong A\mathbb{G} \langle \underbrace{d_1, \dots, d_{n-1}}_{\text{уставлено}}, \beta \mid 2\beta = 0, d_n = \beta - d_1 - \dots - d_{n-1} \rangle \cong \\
&\cong A\mathbb{G} \langle d_1, \dots, d_{n-1}, \beta \mid 2\beta = 0 \rangle \cong \\
&\cong \langle d_1 \mid - \rangle \oplus \dots \oplus \langle d_{n-1} \mid - \rangle \oplus \langle \beta \mid 2\beta = 0 \rangle \cong \\
&\cong \mathbb{Z} \oplus \dots \oplus \mathbb{Z} \oplus \mathbb{Z}_2 \cong \mathbb{Z}^{n-1} \oplus \mathbb{Z}_2
\end{aligned}$$

Комментар: на OT сiao кернелени мултипликативну поведицу,
а слоге агитивну.

$$\begin{array}{ccc}
\text{нпр.} & \langle \beta \mid \beta^2 = 1 \rangle & \cong \langle \beta \mid 2\beta = 0 \rangle \\
& \uparrow & \uparrow \\
& \text{OT} & \text{слоге}
\end{array}$$

$n=2$ Препознати е. у $C_2 = \mathbb{Z}\langle \sigma \rangle$ је $x = k\sigma$, $k \in \mathbb{Z}$

$$x \in \ker d_2 \Leftrightarrow d_2(x) = 0 \Leftrightarrow d_2(k\sigma) = 0 \Leftrightarrow 2k \underbrace{(d_1 + \dots + d_n)}_{\substack{\neq 0 \text{ јер има} \\ \text{неч. број } 0}} = 0 \Leftrightarrow k = 0$$

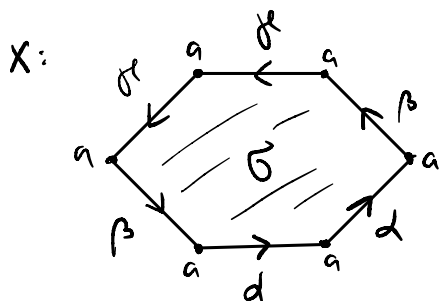
$$\Rightarrow \ker d_2 = 0$$

$$\Rightarrow H_2(N_n) \cong \ker d_2 / \text{im } d_3 = 0/0 = 0$$

$$\text{Закључно, } H_n(N_n) \cong \begin{cases} \mathbb{Z}, & n=0 \\ \mathbb{Z}^{n-1} \oplus \mathbb{Z}_2, & n=1 \\ 0, & \text{иначе} \end{cases}$$

□

6.

0-ken: a 1-ken: d, β, γ 2-ken: σ

$$C_*: 0 \rightarrow \mathbb{Z} \xrightarrow{d_2} \mathbb{Z}^3 \xrightarrow{d_1} \mathbb{Z} \rightarrow 0$$

$\begin{array}{ccc} \parallel & \parallel & \parallel \\ C_2 & C_1 & C_0 \end{array}$

$$\text{inj. } 0 \rightarrow \mathbb{Z}\langle\sigma\rangle \rightarrow \mathbb{Z}\langle d \rangle \oplus \mathbb{Z}\langle\beta\rangle \oplus \mathbb{Z}\langle\gamma\rangle \rightarrow \mathbb{Z}\langle a \rangle \rightarrow 0$$

$$H_n(X) = 0, \quad n \neq 0, 1, 2$$

$n=0$ X je tuzgato noberes $\Rightarrow H_0(X) \cong \mathbb{Z}$

$n=1$ $d_1(d) = d_1(\beta) = d_1(\gamma) = a - a = 0 \Rightarrow d_1 \equiv 0 \Rightarrow \ker d_1 = \mathbb{Z}^3$

$$d_2(\sigma) = d + d + \beta + \gamma + \gamma + \beta = 2(d + \beta + \gamma)$$

$$\Rightarrow \text{im } d_2 = \{d_2(k\sigma) \mid k \in \mathbb{Z}\} = \{k \cdot 2(d + \beta + \gamma) \mid k \in \mathbb{Z}\} = \mathbb{Z}\langle 2(d + \beta + \gamma) \rangle$$

$$H_1(X) \cong \ker d_1 / \text{im } d_2 \cong \frac{\mathbb{Z}\langle d \rangle \oplus \mathbb{Z}\langle \beta \rangle \oplus \mathbb{Z}\langle \gamma \rangle}{\mathbb{Z}\langle 2(d + \beta + \gamma) \rangle}$$

$$= \dots = \mathbb{Z}^2 \oplus \mathbb{Z}_2$$

↑
pauzja kao

ker N_n u zeq. 5

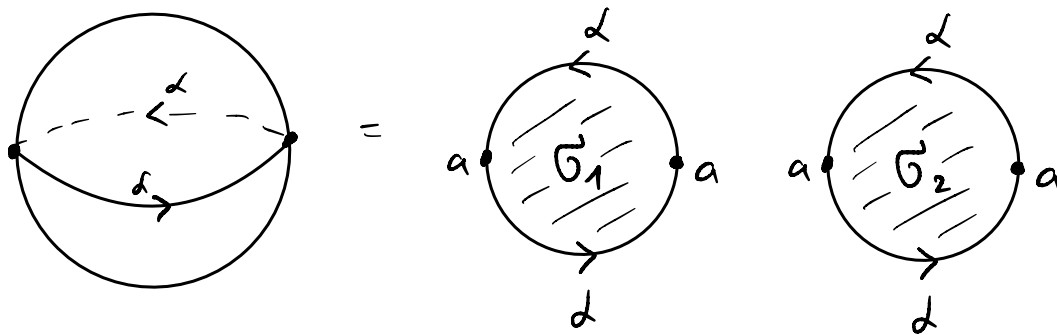
$n=2$ smatramo kao ker N_n : $\ker d_2 = 0 \Rightarrow H_2(X) = 0$

$$\text{Konačno, } H_n(X) \cong \begin{cases} \mathbb{Z}, & n=0 \\ \mathbb{Z}^2 \oplus \mathbb{Z}_2, & n=1 \\ 0, & \text{inače} \end{cases}$$



7.

X:



0-ken: a
 1-ken: d
 2-ken: σ_1, σ_2

$$\Rightarrow C_*: 0 \rightarrow \mathbb{Z}^2 \xrightarrow{d_2} \mathbb{Z} \xrightarrow{d_1} \mathbb{Z} \rightarrow 0$$

$\begin{matrix} \parallel \\ C_2 \end{matrix} \quad \begin{matrix} \parallel \\ C_1 \end{matrix} \quad \begin{matrix} \parallel \\ C_0 \end{matrix}$

$$\text{np. } C_*: 0 \rightarrow \mathbb{Z}\langle\sigma_1\rangle \oplus \mathbb{Z}\langle\sigma_2\rangle \rightarrow \mathbb{Z}\langle d \rangle \rightarrow \mathbb{Z}\langle a \rangle \rightarrow 0$$

$$H_n(X) = 0, \quad n \neq 0, 1, 2$$

$n=0$ X je tujutno noberena $\Rightarrow H_0(X) \cong \mathbb{Z}$

$n=1$ $d_1(d) = a - a = 0 \Rightarrow d_1 \equiv 0 \Rightarrow \ker d_1 = C_1 = \mathbb{Z}$

$$d_2(\sigma_1) = d_2(\sigma_2) = d + d = 2d$$

Preverjamo en. vs $C_2 = \mathbb{Z}\langle\sigma_1\rangle \oplus \mathbb{Z}\langle\sigma_2\rangle$ je $k\sigma_1 + l\sigma_2, k, l \in \mathbb{Z}$

$$\text{im } d_2 = \{d_2(k\sigma_1 + l\sigma_2) \mid k, l \in \mathbb{Z}\} = \{2kd + 2ld \mid k, l \in \mathbb{Z}\} =$$

$$= \{\underbrace{(k+l)}_m 2d \mid k, l \in \mathbb{Z}\} = \{m \cdot 2d \mid m \in \mathbb{Z}\} = \mathbb{Z}\langle 2d \rangle$$

$$\Rightarrow H_1(X) = \ker d_1 / \text{im } d_2 = \mathbb{Z}\langle d \rangle / \mathbb{Z}\langle 2d \rangle = \langle d \mid 2d = 0 \rangle \cong \mathbb{Z}_2$$

$n=2$

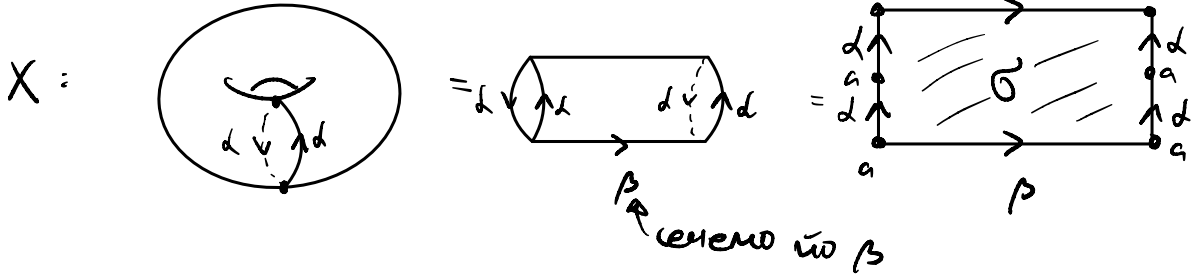
$$x = k\sigma_1 + l\sigma_2 \in \ker d_2 \Leftrightarrow d_2(k\sigma_1 + l\sigma_2) = 0 \Leftrightarrow 2(k+l)d = 0 \Leftrightarrow \underbrace{2(k+l)}_{\neq 0} d = 0 \Leftrightarrow k+l=0 \Leftrightarrow l=-k$$

$$\Rightarrow \ker d_2 = \{k\sigma_1 + l\sigma_2 \mid k, l \in \mathbb{Z}, l = -k\} = \{k\sigma_1 - k\sigma_2 \mid k \in \mathbb{Z}\} \\ = \{k(\sigma_1 - \sigma_2) \mid k \in \mathbb{Z}\} = \mathbb{Z}\langle \sigma_1 - \sigma_2 \rangle$$

$$\Rightarrow H_2(X) \cong \ker d_2 / \operatorname{im} d_1 = \mathbb{Z} / 0 = \mathbb{Z}$$

Котането, $H_n(X) \cong \begin{cases} \mathbb{Z}, & n=0, 2 \\ \mathbb{Z}^2, & n=1 \\ 0, & \text{иначе} \end{cases}$ $\square$

8.



$$\left. \begin{array}{l} 0\text{-ker: } a \\ 1\text{-ker: } d, \beta \\ 2\text{-ker: } \sigma \end{array} \right\} \Rightarrow C_X: \begin{array}{ccccccc} 0 & \rightarrow & \mathbb{Z} & \xrightarrow{d_2} & \mathbb{Z}^2 & \xrightarrow{d_1} & \mathbb{Z} \rightarrow 0 \\ & & \parallel & & \parallel & & \parallel \\ & & C_2 & & C_1 & & C_0 \end{array}$$

и. $0 \rightarrow \mathbb{Z}\langle\sigma\rangle \rightarrow \mathbb{Z}\langle\sigma\rangle \oplus \mathbb{Z}\langle\beta\rangle \rightarrow \mathbb{Z}\langle a\rangle \rightarrow 0$

$$H_n(X) = 0, \quad n \neq 0, 1, 2$$

$n=0$ X је связно пространство $\Rightarrow H_0(X) \cong \mathbb{Z}$

$n=1$ $d_1(d) = d_1(\beta) = a - a = 0 \Rightarrow d_1 = 0 \Rightarrow \ker d_1 = \mathbb{Z}^2$

$$d_2(\sigma) = \beta + d + d - \beta - d - d = 0 \Rightarrow \operatorname{im} d_2 = 0$$

$$\Rightarrow H_1(X) \cong \ker d_1 / \operatorname{im} d_2 = \mathbb{Z}^2 / 0 = \mathbb{Z}^2$$

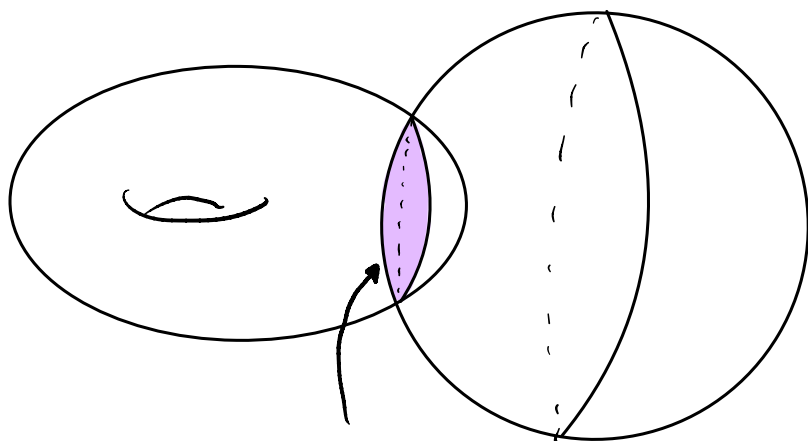
$n=2$ $d_2 = 0 \Rightarrow \ker d_2 = C_2 = \mathbb{Z} \Rightarrow H_2(X) = \ker d_2 / \operatorname{im} d_3 = \mathbb{Z} / 0 = \mathbb{Z}$

Котането, $H_n(X) \cong \begin{cases} \mathbb{Z}, & n=0, 2 \\ \mathbb{Z}^2, & n=1 \\ 0, & \text{иначе} \end{cases}$ $\square$

9. (a) ovoj geo zavrstka je malo tuma, nego mnogo je planirano, pa ga slobodno preokrenite.
Ako nekoj zaima resenje nko mi se javi.

(5)

X:



oznacimo ovaj
podskup sa A

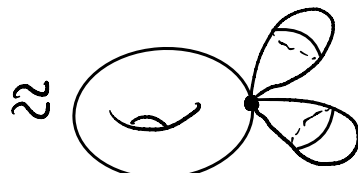
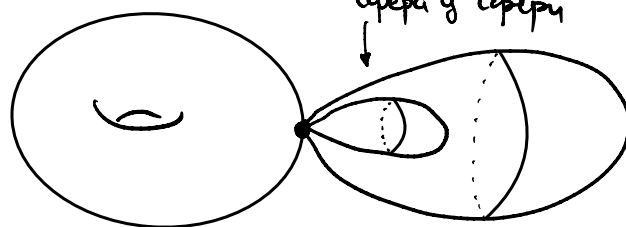
A skupimo u jednu

Primetujemo $A \approx D^2 \approx *$, pa je $X \approx X/A$

↑
disk

↑
isoreno iz OT str. 63

$\Rightarrow X \approx X/A =$



$T^2 \vee S^2 \vee S^2$

$$\Rightarrow \tilde{H}_n(X) \cong \tilde{H}_n(X/A) \cong \tilde{H}_n(T^2) \oplus \tilde{H}_n(S^2) \oplus \tilde{H}_n(S^2) \cong \begin{cases} \mathbb{Z}^2, & n=1 \\ \mathbb{Z}^3, & n=2 \\ 0, & \text{inače} \end{cases}$$

$$\text{tj. } H_n(X) \cong \begin{cases} \mathbb{Z}, & n=0 \\ \mathbb{Z}^2, & n=1 \\ \mathbb{Z}^3, & n=2 \\ 0, & \text{inače} \end{cases}$$

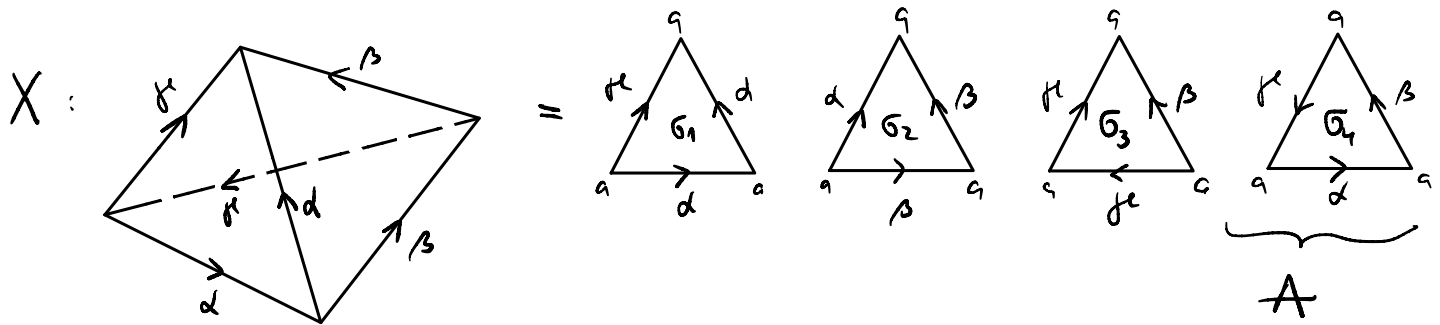
Koristimo smo:

$$H_n(T^2) \cong \begin{cases} \mathbb{Z}, & n=0,2 \\ \mathbb{Z}^2, & n=1 \\ 0, & \text{inače} \end{cases}$$

$$H_n(S^2) = \begin{cases} \mathbb{Z}, & n=0,2 \\ 0, & \text{inače} \end{cases}$$



10.

0-ken: a 1-ken: α, β, γ 2-ken: $\sigma_1, \sigma_2, \sigma_3, \sigma_4$

$$\Rightarrow C_X: 0 \rightarrow \mathbb{Z}^4 \rightarrow \mathbb{Z}^3 \rightarrow \mathbb{Z} \rightarrow 0$$

$$H_n(X) = 0, n \neq 0, 1, 2$$

$m=0$ X je tujutno uobeser $\Rightarrow H_0(X) \cong \mathbb{Z}$

$m=1$ $d_1(\alpha) = d_1(\beta) = d_1(\gamma) = a - a = 0 \Rightarrow d_1 = 0 \Rightarrow \ker d_1 = \mathbb{Z}^3$

$$d_2(\sigma_1) = 2\alpha - \gamma$$

$$d_2(\sigma_2) = 2\beta - \alpha$$

$$d_2(\sigma_3) = \beta - 2\gamma$$

$$d_2(\sigma_4) = \alpha + \beta + \gamma$$

Preuslovani en. γ $C_2 = \mathbb{Z}\langle \sigma_1 \rangle \oplus \mathbb{Z}\langle \sigma_2 \rangle \oplus \mathbb{Z}\langle \sigma_3 \rangle + \mathbb{Z}\langle \sigma_4 \rangle$ je
 $k\sigma_1 + l\sigma_2 + j\sigma_3 + m\sigma_4, k, l, j, m \in \mathbb{Z}$

$$\begin{aligned} d_2(k\sigma_1 + l\sigma_2 + j\sigma_3 + m\sigma_4) &= k(2\alpha - \gamma) + l(2\beta - \alpha) + j(\beta - 2\gamma) + m(\alpha + \beta + \gamma) \\ &= \underbrace{\alpha(2k - l + m)}_a + \underbrace{\beta(2l + j + m)}_b + \underbrace{\gamma(-k - 2j + m)}_c = S \end{aligned}$$

Kaj k, l, j, m upoty $\mathbb{Z}$ u a, b, c upoty $\mathbb{Z}$ $\Rightarrow$ $\text{Im } d_2 = \{a\alpha + b\beta + c\gamma \mid a, b, c \in \mathbb{Z}\} = \mathbb{Z}\langle \alpha \rangle \oplus \mathbb{Z}\langle \beta \rangle \oplus \mathbb{Z}\langle \gamma \rangle = \mathbb{Z}^3$

$$\Rightarrow H_1(X) \cong \ker d_1 / \operatorname{im} d_2 = \mathbb{Z}^3 / \mathbb{Z}^3 = 0$$

$$n=2 \quad \ker d_2 = ?$$

$$k\sigma_1 + l\sigma_2 + j\sigma_3 + m\sigma_4 \in \ker d_2 \Leftrightarrow S = 0$$

$$\Leftrightarrow 2k - l + m = 0$$

$$2l + j + m = 0$$

$$-k - 2j + m = 0$$

решим систему ...

$$\text{решение: } k = -j, l = -j, m = j, \quad j \in \mathbb{Z}$$

$$\Rightarrow \ker d_2 = \{ -j\sigma_1 - j\sigma_2 + j\sigma_3 + j\sigma_4 \mid j \in \mathbb{Z} \}$$

$$= \{ -j(\sigma_1 + \sigma_2 - \sigma_3 - \sigma_4) \mid j \in \mathbb{Z} \}$$

$$= \mathbb{Z} \langle \sigma_1 + \sigma_2 - \sigma_3 - \sigma_4 \rangle$$

$$\Rightarrow H_2(X) \cong \ker d_2 / \operatorname{im} d_3 = \mathbb{Z} / 0 = \mathbb{Z}.$$

$$\text{Итак, } H_n(X) \cong \begin{cases} \mathbb{Z}, & n=0,2 \\ 0, & \text{иначе} \end{cases}$$

(5) $H_0(X, A) = 0$ (т.к. A не содержит компоненты связности топ. гр. X)

$$H_n(X, A) \cong H_n(X/A), \quad n \geq 1$$

$$X/A = \left(\begin{array}{c} \triangle_{\sigma_1} \quad \triangle_{\sigma_2} \quad \triangle_{\sigma_3} \quad \triangle_{\sigma_4} \\ \text{with labels } \alpha, \beta, \gamma, \delta \text{ and arrows} \end{array} \right)$$

$$\approx \begin{array}{c} \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{array} \quad \bullet_a$$

$/A$
кажд A склеиваю у точку
и $\alpha, \beta, \gamma, \delta$ ле склеиве

$$= \text{three-lobed shape} = S^2 \vee S^2 \vee S^2$$

$$\Rightarrow H_n(X, A) \cong H_n(S^2 \vee S^2 \vee S^2) \cong H_n(S^2) \oplus H_n(S^2) \oplus H_n(S^2), n \geq 1$$

$$\Rightarrow H_n(X, A) \cong \begin{cases} \mathbb{Z}^3, & n=2 \\ 0, & \text{иначе} \end{cases}$$

II Найдём: коротчайшим путь от точки a до a (X, A)

$$\dots \rightarrow \tilde{H}_n(A) \rightarrow \tilde{H}_n(X) \rightarrow H_n(X, A) \rightarrow \tilde{H}_{n-1}(A) \rightarrow \dots$$

$$A = \triangle_{\sigma_1} = \triangle_{\sigma_2} = \text{disk with boundary } a \approx \text{disk with } a \text{ expanded} \xrightarrow{\cong} \text{disk with } a \text{ expanded} \xrightarrow{\cong} S^1 \vee S^1$$

расширено a
у g

$$\Rightarrow \tilde{H}_n(A) \cong \begin{cases} \mathbb{Z}^2, & n=1 \\ 0, & \text{иначе} \end{cases}$$

$\Rightarrow$) гл҃вн. нс҃тһе҃р оваро:

$$\begin{array}{ccccccccccc} \tilde{H}_2(A) & & \tilde{H}_2(X) & & & & \tilde{H}_1(A) & & \tilde{H}_1(X) & & & & \tilde{H}_0(A) \\ \text{"} & & \text{"} & & & & \text{"} & & \text{"} & & & & \text{"} \\ \cdots \rightarrow 0 \rightarrow \mathbb{Z} \rightarrow H_2(X, A) \rightarrow \mathbb{Z}^2 \rightarrow 0 \rightarrow H_1(X, A) \rightarrow 0 \rightarrow \cdots \end{array}$$

← все нуле
наверху

→ все нуле
наверху

из этой дилл. можно кратко п.н.

$$0 \rightarrow \mathbb{Z} \rightarrow H_2(X, \mathbb{A}) \rightarrow \mathbb{Z}^2 \rightarrow 0 \quad \sim \quad 0 \rightarrow H_1(X, \mathbb{A}) \rightarrow 0$$

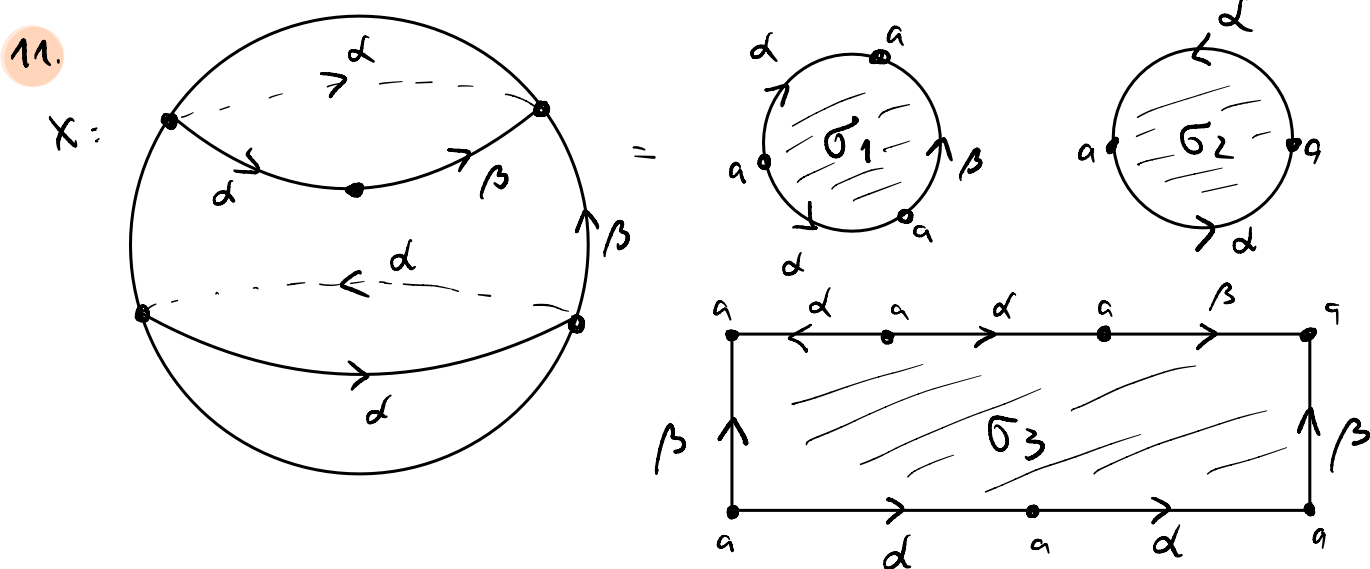
$\uparrow$ $\Downarrow$
 свободные $H_1(X, \mathbb{A}) = 0$

$\Rightarrow$ laag kwik. te zien, na je

$$H_2(X, \mathbb{A}) \cong \mathbb{Z} \oplus \mathbb{Z}^2 \cong \mathbb{Z}^3$$

(штаб на шр. 9 у бележ~~ка~~ма)

Конечно, $H_n(X, A) \cong \begin{cases} \mathbb{Z}^3, n=2 \\ 0, \text{ иначе} \end{cases}$



O-ken: a

1-ken: $d_{1\beta}$

2- her: $\sigma_1, \sigma_2, \sigma_3$

$$\Rightarrow C_*: 0 \rightarrow \mathbb{Z} \xrightarrow{d_2} \mathbb{Z}^2 \xrightarrow{d_1} \mathbb{Z} \rightarrow 0$$

$n=0$ X је тачно повесан $\Rightarrow H_0(X) \cong \mathbb{Z}$

$$n=1 \quad d_1(d) = d_1(\beta) = 9 - 9 = 0 \Rightarrow d_1 = 0 \Rightarrow \ker d_1 = \mathbb{Z}^2$$

$$d_2(\sigma_1) = \alpha + \beta - \alpha = \beta$$

$$d_2(G_2) = d + d = 2d$$

$$d_2(\sigma_3) = \alpha + \alpha + \beta - \beta - \alpha + \alpha - \beta = 2\alpha - \beta$$

Пространств ел. γ $C_2 = \mathbb{Z}\langle \sigma_1 \rangle \oplus \mathbb{Z}\langle \sigma_2 \rangle \oplus \mathbb{Z}\langle \sigma_3 \rangle$ $\bar{f} \in$
 образа $k\sigma_1 + l\sigma_2 + m\sigma_3, k, l, m \in \mathbb{Z}$ $\subset$

$$d_2(kG_1 + lG_2 + mG_3) = k\beta + 2l\alpha + m(2\alpha - \beta) = \overbrace{\alpha \cdot 2(l+m) + \beta(-k-m)}$$

$$\Rightarrow \text{imd}_2 = \left\{ \alpha \cdot \underbrace{2(\ell+m)}_a + \underbrace{\beta(k-m)}_b \mid k, \ell, m \in \mathbb{Z} \right\} = \{ 2a\alpha + 6\beta \mid a, b \in \mathbb{Z} \}$$

$$= \pi \langle 2\alpha \rangle \oplus \pi \langle \beta \rangle$$

$$\Rightarrow H_1(X) = \ker d_1 / \operatorname{im} d_2 \cong \mathbb{Z}\langle \alpha \rangle \oplus \mathbb{Z}\langle \beta \rangle / \mathbb{Z}\langle 2\alpha \rangle \oplus \mathbb{Z}\langle \beta \rangle$$

$$\cong \mathbb{Z}\langle \alpha, \beta \mid 2\alpha = 0, \beta = 0 \rangle = \langle \alpha \mid 2\alpha = 0 \rangle = \mathbb{Z}_2$$

$$n=2 \quad kG_1 + lG_2 + mG_3 \in \ker d_2 \Leftrightarrow S = 0$$

$$\Leftrightarrow 2(l+m) = 0 \quad \Leftrightarrow k = m, \quad l = -m, \quad m \in \mathbb{Z}$$

$$k - m = 0$$

$$\Rightarrow \ker d_2 = \{ m\sigma_1 - m\sigma_2 + m\sigma_3 \mid m \in \mathbb{Z} \} = \{ m(\sigma_1 - \sigma_2 + \sigma_3) \mid m \in \mathbb{Z} \} \cong \mathbb{Z}$$

$$\Rightarrow H_2(X) \cong \ker d_2 / \operatorname{im} d_3 = \mathbb{Z} / 0 = \mathbb{Z}$$

Контину, $H_n(X) \cong \begin{cases} \mathbb{Z}, & n=0,2 \\ \mathbb{Z}_2, & n=1 \\ 0, & \text{иначе} \end{cases}$

Сей же определим $H_n(X; \mathbb{Z}_2)$.

ТОУК: $H_n(X; \mathbb{Z}_2) \cong (H_n(X) \otimes \mathbb{Z}_2) \oplus \operatorname{Tor}(H_{n-1}(X), \mathbb{Z}_2)$
(упр. 36)

$H_n(X)$ и $H_{n-1}(X)$ нуль за $n \neq 0, 1, 2, 3$ и в этих случаях $H_n(X; \mathbb{Z}_2) = 0$
Тогда $n = 0, 1, 2, 3$.

$$H_0(X; \mathbb{Z}_2) \cong H_0(X) \otimes \mathbb{Z}_2 \oplus \operatorname{Tor}(0, \mathbb{Z}_2) \cong \mathbb{Z} \otimes \mathbb{Z}_2 \cong \mathbb{Z}_2$$

$$H_1(X; \mathbb{Z}_2) \cong H_1(X) \otimes \mathbb{Z}_2 \oplus \operatorname{Tor}(H_0(X), \mathbb{Z}_2) \cong \underbrace{\mathbb{Z}_2 \otimes \mathbb{Z}_2}_{\mathbb{Z}_2} \oplus \underbrace{\operatorname{Tor}(\mathbb{Z}, \mathbb{Z}_2)}_0 = \mathbb{Z}_2$$

$$H_2(X; \mathbb{Z}_2) \cong H_2(X) \otimes \mathbb{Z}_2 \oplus \operatorname{Tor}(H_1(X), \mathbb{Z}_2)$$

$$\cong \mathbb{Z} \otimes \mathbb{Z}_2 \oplus \operatorname{Tor}(\mathbb{Z}_2, \mathbb{Z}_2) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$$

$$H_3(X; \mathbb{Z}_2) \cong 0 \otimes \mathbb{Z}_2 \oplus \operatorname{Tor}(\mathbb{Z}, \mathbb{Z}_2) = 0$$

Контину, $H_n(X; \mathbb{Z}_2) \cong \begin{cases} \mathbb{Z}_2, & n=0,1 \\ \mathbb{Z}_2^2, & n=2 \\ 0, & \text{иначе} \end{cases}$

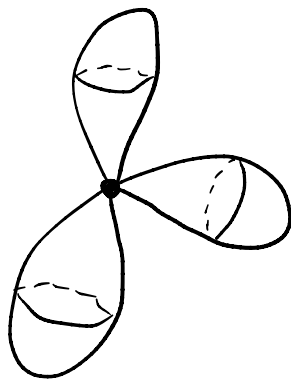
(5) доп. лемма о компонентах нулевой гомологии X

$$\Rightarrow H_0(X, \alpha \cup \beta) = 0$$

$$\text{За } n \geq 1 \text{ имеем } H_n(X, \alpha \cup \beta) \cong H_n(X / (\alpha \cup \beta))$$

$$X/(\alpha \cup \beta) \cong$$

↑
связано α и β
и много



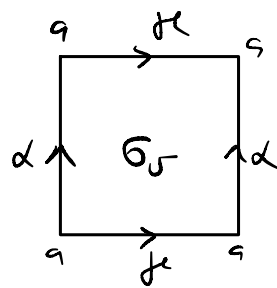
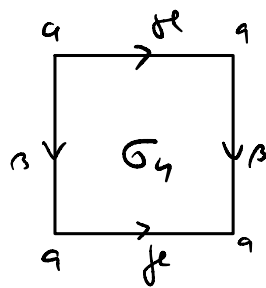
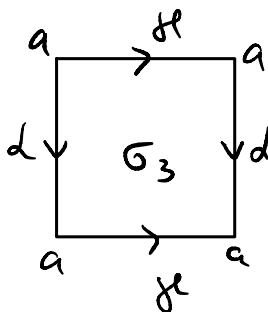
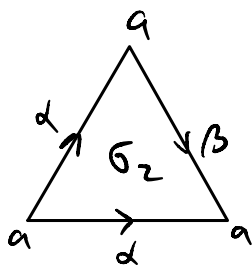
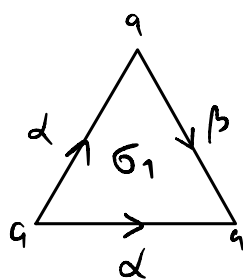
$$= S^2 \vee S^2 \vee S^2$$

$$\Rightarrow H_n(X, \alpha \cup \beta) \cong H_n(S^2 \vee S^2 \vee S^2) \cong H_n(S^2) \oplus H_n(S^2) \oplus H_n(S^2)$$

Результат, $H_n(X, \alpha \cup \beta) \cong \begin{cases} \mathbb{Z}^3, & n=2 \\ 0, & \text{иначе} \end{cases}$ $\square$

12.

X:



0-клетка: a

1-клетка: α, β, γ

2-клетка: $\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5$

$$\Rightarrow C_*: 0 \rightarrow \underset{C_2}{\mathbb{Z}} \xrightarrow{d_2} \underset{C_1}{\mathbb{Z}} \xrightarrow{d_1} \underset{C_0}{\mathbb{Z}} \rightarrow 0$$

$$n \neq 0, 1, 2 \Rightarrow H_n(X) = 0$$

$n=0$ X је связно пространство $\Rightarrow H_0(X) \cong \mathbb{Z}$

$n=1$ $d_1(\alpha) = d_1(\beta) = d_1(\gamma) = a - a = 0 \Rightarrow d_1 = 0 \Rightarrow \ker d_1 = \mathbb{Z}^3$

$$d_2(\sigma_1) = \alpha - \beta - \alpha = -\beta$$

$$d_2(\sigma_2) = \alpha - \beta - \alpha = -\beta$$

$$d_2(\sigma_3) = \gamma - \alpha - \gamma + \alpha = 0$$

$$d_2(\sigma_4) = 0$$

$$d_2(\sigma_5) = 0$$

$$d_2(k\sigma_1 + l\sigma_2 + m\sigma_3 + p\sigma_4 + q\sigma_5) = -k\beta - l\beta = -\beta(k+l)$$

$$\Rightarrow \text{im } d_2 = \{ -\beta(k+l) \mid k, l \in \mathbb{Z} \} = \{ b\beta \mid b \in \mathbb{Z} \} = \mathbb{Z}\langle \beta \rangle$$

$$\Rightarrow H_1(X) \cong \ker d_1 / \text{im } d_2 = \mathbb{Z}\langle \alpha \rangle \oplus \mathbb{Z}\langle \beta \rangle \oplus \mathbb{Z}\langle \gamma \rangle / \mathbb{Z}\langle \beta \rangle \cong \mathbb{Z}^2$$

$$n=2 \quad k\sigma_1 + l\sigma_2 + m\sigma_3 + p\sigma_4 + q\sigma_5 \in \ker d_2$$

$$\Leftrightarrow -\beta(k+l) = 0 \Leftrightarrow l = -k$$

$$\begin{aligned} \Rightarrow \ker d_2 &= \{ k\sigma_1 - k\sigma_2 + m\sigma_3 + p\sigma_4 + q\sigma_5 \mid k, m, p, q \in \mathbb{Z} \} \\ &= \{ k(\sigma_1 - \sigma_2) + m\sigma_3 + p\sigma_4 + q\sigma_5 \mid k, m, p, q \in \mathbb{Z} \} \\ &= \mathbb{Z}\langle \sigma_1 - \sigma_2 \rangle \oplus \mathbb{Z}\langle \sigma_3 \rangle \oplus \mathbb{Z}\langle \sigma_4 \rangle \oplus \mathbb{Z}\langle \sigma_5 \rangle \\ &= \mathbb{Z}^4 \end{aligned}$$

$$\Rightarrow H_2(X) = \ker d_2 / \text{im } d_3 = \mathbb{Z}^4 / 0 = \mathbb{Z}^4$$

$$\Rightarrow H_n(X) \cong \begin{cases} \mathbb{Z}, & n=0 \\ \mathbb{Z}^2, & n=1 \\ \mathbb{Z}^4, & n=2 \\ 0, & \text{otherwise} \end{cases}$$

$$H_n(X; \mathbb{Z}_3) = 0 \quad \text{for } n \neq 0, 1, 2, 3$$

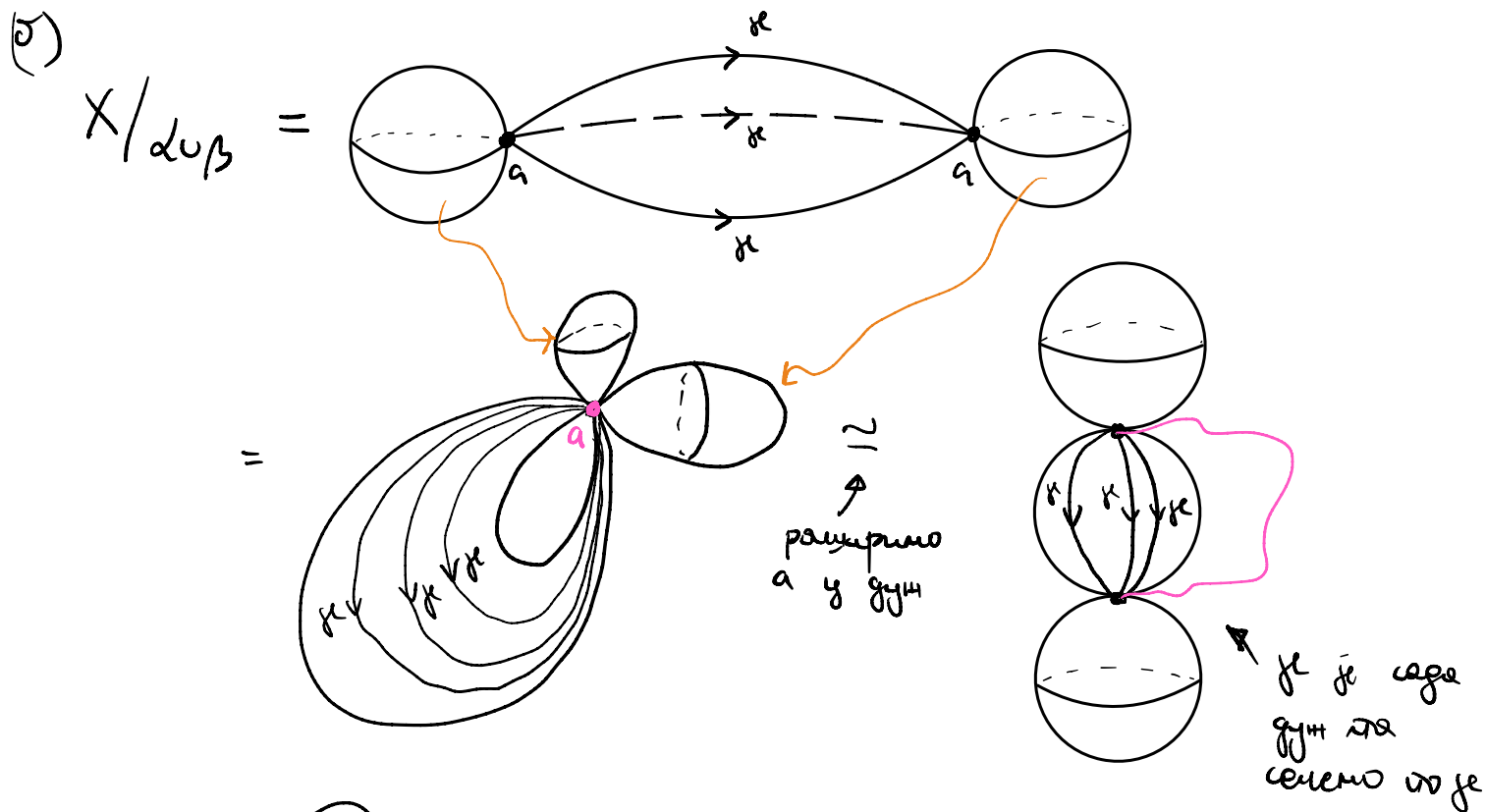
$$H_0(X; \mathbb{Z}_3) = \mathbb{Z}_3$$

$$\begin{aligned} H_1(X; \mathbb{Z}_3) &= (H_1(X) \otimes \mathbb{Z}_3) \oplus \text{Tor}(H_0(X), \mathbb{Z}_3) \\ &= (\mathbb{Z}^2 \otimes \mathbb{Z}_3) \oplus 0 = \mathbb{Z}_3^2 \end{aligned}$$

$$H_2(X; \mathbb{Z}_3) = (\mathbb{Z}^4 \otimes \mathbb{Z}_3) \oplus \underbrace{\text{Tor}(\mathbb{Z}^2, \mathbb{Z}_3)}_0 = \mathbb{Z}_3^4$$

$$H_3(X; \mathbb{Z}_3) = 0$$

$$\Rightarrow H_n(X; \mathbb{Z}_3) = \begin{cases} \mathbb{Z}_3, & n=0 \\ \mathbb{Z}_3^2, & n=1 \\ \mathbb{Z}_3^4, & n=2 \\ 0, & \text{otherwise} \end{cases}$$



12
1je

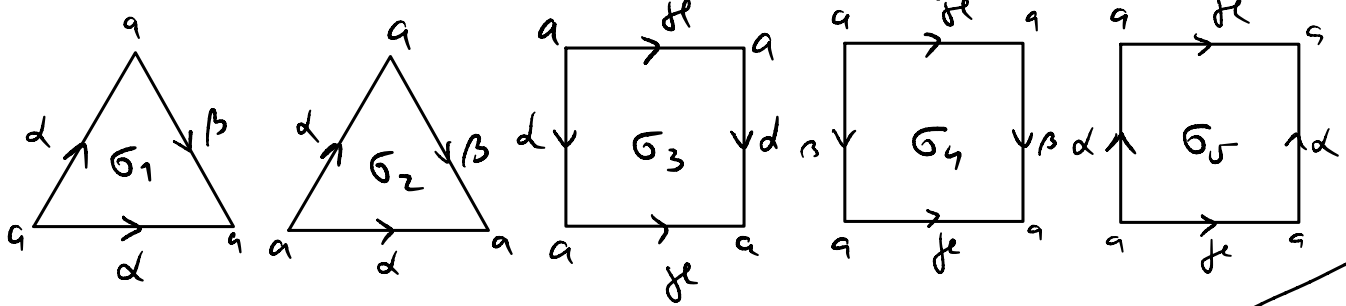
$$= S^1 \vee S^2 \vee S^2 \vee S^2 \vee S^2 \vee S^2$$

$$\Rightarrow H_n(X, d\alpha\beta) \cong H_n(S^1 \vee S^2 \vee S^2 \vee S^2 \vee S^2 \vee S^2), \quad n \geq 1$$

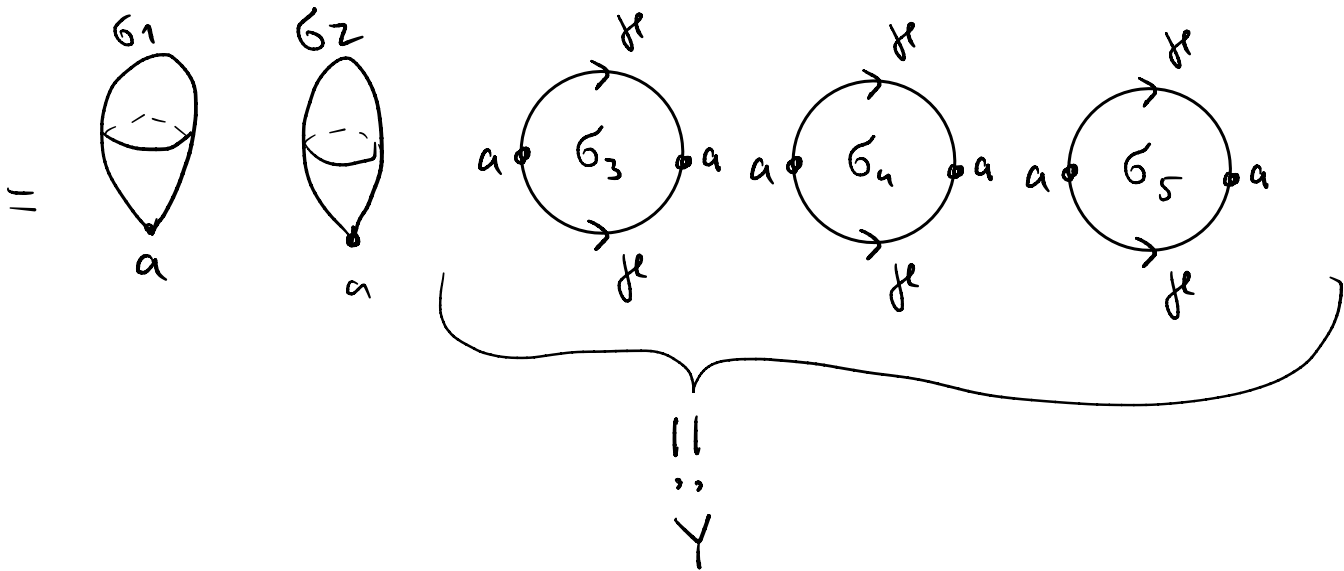
$$\Rightarrow H_n(X, d\alpha\beta) \cong \begin{cases} \mathbb{Z}, & n=1 \\ \mathbb{Z}^5, & n=5 \\ 0, & \text{otherwise} \end{cases}$$

II Найдем: сечение γ равносторонней мореры:

X :



$\alpha \cup \beta$



$$\Rightarrow X / \alpha \cup \beta = Y \vee S^2 \vee S^2$$

Определим посылку $H_n(Y)$ и еще укажем γ

$$H_n(X, \alpha \cup \beta) \cong H_n(Y) \oplus H_n(S^2) \oplus H_n(S^2) \dots$$



13. (a) Копутируемо ТДГК

$$H_n(X; \mathbb{Z}_5) \cong \left(H_n(X) \otimes \mathbb{Z}_5 \right) \oplus \text{Tor}(H_{n-1}(X), \mathbb{Z}_5)$$

Оскільки $H_n(X; \mathbb{Z}_5) = 0, n \geq 6$

$$H_0(X; \mathbb{Z}_5) \cong \mathbb{Z} \otimes \mathbb{Z}_5 \cong \mathbb{Z}_5$$

$$H_1(X; \mathbb{Z}_5) \cong \left((\mathbb{Z}_4 \oplus \mathbb{Z}_5) \otimes \mathbb{Z}_5 \right) \oplus \text{Tor}(\mathbb{Z}, \mathbb{Z}_5) \cong \mathbb{Z}_5$$

$$H_2(X; \mathbb{Z}_5) \cong (0 \otimes \mathbb{Z}_5) \oplus \text{Tor}(\mathbb{Z}_4 \oplus \mathbb{Z}_5, \mathbb{Z}_5) \cong \mathbb{Z}_5$$

$$H_3(X; \mathbb{Z}_5) \cong (0 \otimes \mathbb{Z}_5) \oplus \text{Tor}(0, \mathbb{Z}_5) = 0$$

$$H_4(X; \mathbb{Z}_5) \cong (\mathbb{Z} \otimes \mathbb{Z}_5) \oplus \text{Tor}(0, \mathbb{Z}_5) \cong \mathbb{Z}_5$$

$$H_5(X; \mathbb{Z}_5) \cong (0 \otimes \mathbb{Z}_5) \oplus \text{Tor}(\mathbb{Z}, \mathbb{Z}_5) = 0$$

(б) ТДГК:

$$H_4(X \times S^3; \mathbb{Z}_2) \cong \left(H_4(X \times S^3) \otimes \mathbb{Z}_2 \right) \oplus \text{Tor}(H_3(X \times S^3), \mathbb{Z}_2)$$

Щоб отримати кінцеве рівняння порівняємо $H_4(X \times S^3)$ з $H_3(X \times S^3)$

$$\begin{aligned} H_3(X \times S^3) &\cong (H_0(X) \otimes H_3(S^3)) \oplus (H_1(X) \otimes H_2(S^3)) \oplus (H_2(X) \otimes H_1(S^3)) \\ &\quad \oplus (H_3(X) \otimes H_0(S^3)) \oplus \text{Tor}(H_0(X), H_2(S^3)) \oplus \text{Tor}(H_1(X), H_1(S^3)) \\ &\quad \oplus \text{Tor}(H_2(X), H_0(S^3)) = \\ &\cong (\mathbb{Z} \otimes \mathbb{Z}) \oplus ((\mathbb{Z}_4 \oplus \mathbb{Z}_5) \otimes 0) \oplus (0 \otimes 0) \\ &\quad \oplus (0 \otimes \mathbb{Z}) \oplus \text{Tor}(\mathbb{Z}, 0) \oplus \text{Tor}(0, 0) \oplus \text{Tor}(0, \mathbb{Z}) \\ &\cong \mathbb{Z} \end{aligned}$$

Саме так:

$$\begin{aligned} H_4(X \times S^3) &\cong (\mathbb{Z} \otimes 0) \oplus ((\mathbb{Z}_4 \oplus \mathbb{Z}_5) \otimes \mathbb{Z}) \oplus (0 \otimes 0) \oplus (0 \otimes 0) \oplus (\mathbb{Z} \otimes \mathbb{Z}) \\ &\quad \oplus \text{Tor}(\mathbb{Z}, \mathbb{Z}) \oplus \text{Tor}(\mathbb{Z}_4 \oplus \mathbb{Z}_5, 0) \oplus \text{Tor}(0, 0) \oplus \text{Tor}(0, \mathbb{Z}) \\ &\cong \mathbb{Z}_4 \oplus \mathbb{Z}_5 \oplus \mathbb{Z} \end{aligned}$$

$$\begin{aligned}
\Rightarrow H_4(X \times S^3; \mathbb{Z}_2) &\cong (H_4(X \times S^3) \otimes \mathbb{Z}_2) \oplus \text{Tor}(H_3(X \times S^3), \mathbb{Z}_2) \\
&\cong \left((\mathbb{Z}_4 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}) \otimes \mathbb{Z}_2 \right) \oplus \underbrace{\text{Tor}(\mathbb{Z}, \mathbb{Z}_2)}_0 \\
&\cong \underbrace{(\mathbb{Z}_4 \otimes \mathbb{Z}_2)}_{\mathbb{Z}_2} \oplus \underbrace{(\mathbb{Z}_5 \otimes \mathbb{Z}_2)}_0 \oplus \underbrace{(\mathbb{Z} \otimes \mathbb{Z}_2)}_{\mathbb{Z}_2} \cong \mathbb{Z}_2^2 \quad \square
\end{aligned}$$

15. Вспомогательное утверждение $H_n(S^2 \times S^3)$ по теореме Куннета для

$$H_i(S^2) = 0, i \geq 3 \text{ и } H_j(S^3) = 0 \text{ за } j \geq 4$$

$$\Rightarrow H_n(S^2 \times S^3) = 0 \text{ за } n \geq 7$$

$$H_0(S^2 \times S^3) \cong \mathbb{Z}$$

$$H_1(S^2 \times S^3) \cong (\mathbb{Z} \otimes 0) \oplus (0 \otimes \mathbb{Z}) \oplus \text{Tor}(\mathbb{Z}, \mathbb{Z}) = 0$$

$$H_2(S^2 \times S^3) \cong (\mathbb{Z} \otimes 0) \oplus (0 \otimes 0) \oplus (\mathbb{Z} \otimes \mathbb{Z}) \oplus \text{Tor}(\mathbb{Z}, 0) \oplus \text{Tor}(0, \mathbb{Z}) \cong \mathbb{Z}$$

$$H_3(S^2 \times S^3) \cong (\mathbb{Z} \otimes \mathbb{Z}) \oplus (0 \otimes 0) \oplus (\mathbb{Z} \otimes 0) \oplus (0 \otimes \mathbb{Z}) \oplus 0 \oplus 0 \oplus 0 \cong \mathbb{Z}$$

(здесь Tor из-за того, что 0 есть $H_i(S^2)$ и $H_i(S^3)$ или 0 или $\mathbb{Z}$, а $\mathbb{Z}$ не свободен)

$$H_4(S^2 \times S^3) \cong (\mathbb{Z} \otimes 0) \oplus (0 \otimes \mathbb{Z}) \oplus (\mathbb{Z} \otimes 0) \oplus (0 \otimes 0) \oplus (0 \otimes \mathbb{Z}) = 0$$

$$H_5(S^2 \times S^3) \cong (\mathbb{Z} \otimes 0) \oplus (0 \otimes 0) \oplus (\mathbb{Z} \otimes \mathbb{Z}) \oplus (0 \otimes 0) \oplus (0 \otimes 0) \oplus (0 \otimes \mathbb{Z}) \cong \mathbb{Z}$$

$$H_6(S^2 \times S^3) = 0$$

$$\Rightarrow H_n(S^2 \times S^3) \cong \begin{cases} \mathbb{Z}, & n=0, 2, 3, 5 \\ 0, & \text{иначе} \end{cases}$$

$$\begin{aligned}
&\xrightarrow{\text{Торик}} \\
&\dots \quad H_n(S^2 \times S^3; \mathbb{Z}_5) \cong \begin{cases} \mathbb{Z}_5, & n=0, 2, 3, 5 \\ 0, & \text{иначе} \end{cases} \quad \square
\end{aligned}$$