(ЈАН 2)

$\boxed{10.}$ (а) $\displaystyle\int_{-\pi}^{\pi} \frac{\sin x + x + 3}{1+|x|}\,dx$; (б) $\displaystyle\int_{-1}^{1} \frac{\sin x \cos x - |x| + 3\operatorname{arctg} x}{(1+x^2)^{\frac{3}{2}}}\,dx$

решење

$$(а)\quad \int_{-\pi}^{\pi} \frac{\sin x + x + 3}{1+|x|}\,dx = \int_{-\pi}^{\pi} \overbrace{\frac{\sin x + x}{1+|x|}}^{\text{непарна}}\,dx + \int_{-\pi}^{\pi} \overbrace{\frac{3}{1+|x|}}^{\text{парна}}\,dx =$$

$$= 0 + 2\int_{0}^{\pi} \frac{3}{1+|x|}\,dx = 2\int_{0}^{\pi} \frac{3}{1+x}\,dx = 6\ln|1+x|\Big|_{0}^{\pi} = 6\ln(1+\pi)$$

(б) $\displaystyle\int_{-1}^{1} \frac{\sin x\cos x - |x| + 3\,\mathrm{arctg}\, x}{(1+x^2)^{\frac{3}{2}}}\,dx = \underbrace{\int_{-1}^{1} \frac{\sin x\cos x + 3\,\mathrm{arctg}\, x}{(1+x^2)^{\frac{3}{2}}}\,dx}_{\boxed{\text{непарна}}} -$

$$- \underbrace{\int_{-1}^{1} \frac{|x|}{(1+x^2)^{\frac{3}{2}}}\,dx}_{\boxed{\text{парна}}} = -2\int_0^1 \frac{x}{(1+x^2)^{\frac{3}{2}}} = \begin{bmatrix} t = 1+x^2 \\ \frac{1}{2}\,dt = x\,dx \\ x=0 \to t=1 \\ x=1 \to t=2 \end{bmatrix} =$$

$$= -\int_1^2 \frac{dt}{t^{\frac{3}{2}}} = \frac{2}{\sqrt{t}}\Bigg|_1^2 = \frac{2}{\sqrt{2}} - \frac{2}{\sqrt{1}} = \sqrt{2} - 2 \qquad \blacksquare$$

11. (а) $\displaystyle\int_0^{30\pi} \sqrt{1-\cos 2x}\,dx$; (б) $\displaystyle\int_0^{\pi} \frac{\sin^3 x\cos x}{1+\cos^2 x}\,dx$;

(в) $\displaystyle\int_{-11\pi}^{20\pi} \mathrm{sgn}\,(\sin x)\,dx$

решење

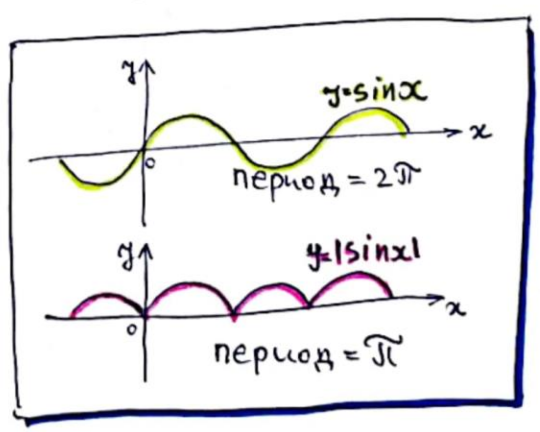
(а) $\displaystyle\int_0^{30\pi} \sqrt{1-\cos 2x}\,dx = \int_0^{30\pi} \sqrt{2\sin^2 x}\,dx = \sqrt{2}\int_0^{30\pi} |\sin x|\,dx =$

$$= \sqrt{2}\cdot 30\cdot \int_0^{\pi} |\sin x|\,dx =$$

$$= 30\sqrt{2}\int_0^{\pi} \sin x\,dx =$$

$$= -30\sqrt{2}\,\cos x\Big|_0^{\pi} = -30\sqrt{2}\underbrace{\cos\pi}_{-1} +$$

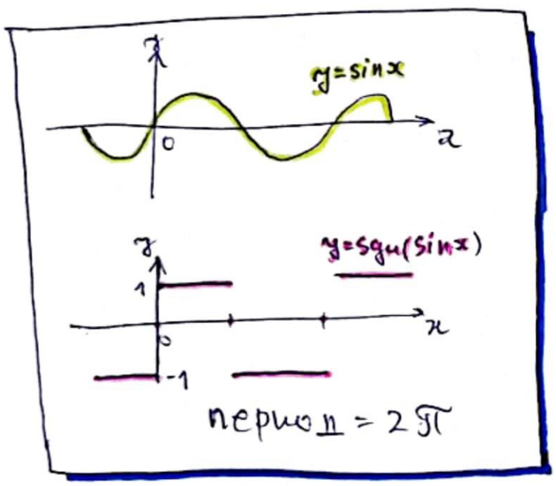
$$+ 30\sqrt{2}\underbrace{\cos 0}_{1} = 60\sqrt{2}$$



(б) $\displaystyle\int_0^{\pi} \frac{\sin^3 x\cos x}{1+\cos^2 x}\,dx = \left[\begin{array}{l} t = x-\frac{\pi}{2} \\ dt = dx \\ x=0 \to t=-\frac{\pi}{2} \\ x=\frac{\pi}{2} \to t=\frac{\pi}{2} \end{array}\right] = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\sin^3(t+\frac{\pi}{2})\cos(t+\frac{\pi}{2})}{1+\cos^2(t+\frac{\pi}{2})}\,dt =$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \underbrace{\frac{\cos^3 t\,(-\sin t)}{1+\sin^2 t}}_{\text{Непарна}}\,dt = 0$$

(в) $\displaystyle\int_{-11\pi}^{20\pi} sgn(\sin x)\,dx = \underbrace{\int_{-11\pi}^{11\pi} sgn(\sin x)\,dx}_{\text{Непарна}} + \int_{11\pi}^{20\pi} sgn(\sin x)\,dx =$



$$= \int_{11-8\cdot 2\pi}^{20-8\cdot 2\pi} sgn(\sin x)\,dx = \int_{-5\pi}^{4\pi} sgn(\sin x)\,dx =$$

$$= \int_{-5\pi}^{-4\pi} sgn(\sin x)\,dx + \underbrace{\int_{-4\pi}^{4\pi} sgn(\sin x)\,dx}_{0} =$$

$$= \int_{\pi}^{2\pi} sgn(\sin x)\,dx = -\int_{\pi}^{2\pi} dx = -x\Big|_{\pi}^{2\pi} = -\pi$$

12. (а) $\displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{x+\cos x}{3|\sin x|+2\cos^2 x}\,dx$; (б) $\displaystyle\int_0^{\pi} \frac{\sin^3 x\cdot|\cos x|}{1+\cos^2 x}\,dx$

решение

(а) $\displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{x+\cos x}{3|\sin x|+2\cos^2 x}\,dx = \underbrace{\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{x\,dx}{3|\sin x|+2\cos^2 x}}_{\text{Непарна}} + \underbrace{\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cos x}{3|\sin x|+2\cos^2 x}\,dx}_{\text{парна}} =$

$$= 2\int_0^{\frac{\pi}{2}} \frac{\cos x}{3\sin x + 2\cos^2 x}\,dx = 2\int_0^{\frac{\pi}{2}} \frac{\cos x}{3\sin x + 2 - 2\sin^2 x}\,dx =$$

$$= \left[\begin{array}{l} t = \sin x \\ dt = \cos x\,dx \\ x = 0 \rightarrow t = 0 \\ x = \frac{\pi}{2} \rightarrow t = 1 \end{array}\right] = 2\int_0^1 \frac{dt}{3t + 2 - 2t^2} = 2\int_0^1 \frac{dt}{(2-t)(2t+1)} = \text{(A)}$$

$$\frac{1}{(2-t)(2t+1)} = \frac{A}{2-t} + \frac{B}{2t+1} \Big/ \cdot (2-t)(2t+1)$$

$$1 = 2At + A + 2B - Bt$$

$$1 = t(2A - B) + A + 2B$$

$$\left.\begin{array}{ll} \text{уз } t: & 0 = 2A - B \\ \text{уз } 1: & 1 = A + 2B \end{array}\right\} \Rightarrow A = \frac{1}{5},\ B = \frac{2}{5}$$

$$\text{(A)} = \frac{2}{5}\int_0^1 \frac{dt}{2-t} + \frac{4}{5}\int_0^1 \frac{dt}{2t+1} = -\frac{2}{5}\ln|2-t| + \frac{2}{5}\ln|2t+1|\Big|_0^1$$

$$= \frac{2}{5}\ln\left|\frac{2t+1}{2-t}\right|\Bigg|_0^1 = \frac{2}{5}\ln 3 - \frac{2}{5}\ln\frac{1}{2} = \frac{2}{5}\ln 6$$

(б) $$\int_0^{\pi} \frac{\sin^3 x \cdot |\cos x|}{1 + \cos^2 x}\,dx = \left[\begin{array}{l} t = x - \frac{\pi}{2} \\ dt = dx \\ x = 0 \rightarrow t = -\frac{\pi}{2} \\ x = \pi \rightarrow t = \frac{\pi}{2} \end{array}\right] = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\sin^3(t + \frac{\pi}{2}) \cdot |\cos(t + \frac{\pi}{2})|}{1 + \cos^2(t + \frac{\pi}{2})}\,dt =$$

$\boxed{|\sin t| = \sin t \text{ на } [0, \frac{\pi}{2}]}$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cos^3 t \cdot |-\sin t|}{1 + (-\sin t)^2}\,dt = 2\int_0^{\frac{\pi}{2}} \frac{\cos^3 t \cdot \sin t}{2 - \cos^2 t}\,dt =$$

$\boxed{\text{парна}}$

$$= \left[\begin{array}{l} p = \cos t \\ -dp = \sin t\,dt \\ t=0 \to p=1 \\ t=\frac{\pi}{2} \to p=0 \end{array}\right] = -2\int_1^0 \frac{p^3}{2-p^2}\,dp = 2\int_0^1 \frac{p^3-2p+2p}{2-p^2}\,dp =$$

$$= 2\int_0^1 \left(-p + \frac{2p}{2-p^2}\right) dp = -p^2\Big|_0^1 + 2\int_0^1 \frac{2p}{2-p^2}\,dp =$$

$$= \left[\begin{array}{l} y = 2-p^2 \\ -dy = 2p\,dp \\ p=0 \to y=2 \\ p=1 \to y=1 \end{array}\right] = -1 - 2\int_2^1 \frac{dy}{y} = -1 + 2\int_1^2 \frac{dy}{y} =$$

$$= -1 + (2\ln|y|)\Big|_1^2 = 2\ln 2 - 1 \quad \blacksquare$$

13. Ако је f непрекидна, доказати:

(а) $\displaystyle\int_0^{\frac{\pi}{2}} f(\sin x)\,dx = \int_0^{\frac{\pi}{2}} f(\cos x)\,dx$;

(б) $\displaystyle\int_0^{\pi} x f(\sin x)\,dx = \frac{\pi}{2}\int_0^{\pi} f(\sin x)\,dx$.

решење

(а) $$\int_0^{\frac{\pi}{2}} f(\sin x)\,dx = \left[\begin{array}{l} t = \frac{\pi}{2} - x \\ dt = -dx \\ x=0 \to t=\frac{\pi}{2} \\ x=\frac{\pi}{2} \to t=0 \end{array}\right] = -\int_{\frac{\pi}{2}}^0 f\left(\sin\left(\frac{\pi}{2}-x\right)\right) dx =$$

$$= \int_0^{\frac{\pi}{2}} f(\cos x)\,dx$$

(б) $$\underbrace{\int_0^{\pi} x f(\sin x)\,dx}_{I} = \left[\begin{array}{l} x=\pi-t \\ dx=-dt \\ x=0 \to t=\pi \\ x=\pi \to t=0 \end{array}\right] = -\int_{\pi}^{0} (\pi-t) f(\sin(\pi-t))\,dt$$

$$= \int_0^{\pi} (\pi-t) f(\sin t)\,dt = \pi\int_0^{\pi} f(\sin t)\,dt - \underbrace{\int_0^{\pi} t f(\sin t)\,dt}_{I}$$

$$\Rightarrow \quad I = \pi\int_0^{\pi} f(\sin t)\,dt - I$$

$$\Rightarrow \quad I = \frac{\pi}{2}\int_0^{\pi} f(\sin t)\,dt \quad \blacksquare$$

14. (а) $\displaystyle\int_0^{\frac{\pi}{4}} \ln(1+\operatorname{tg} x)\,dx$; (б) $\displaystyle\int_0^{1} \frac{\cos^2\frac{\pi x}{2}}{\sqrt{x-x^2}}\,dx$

решење

(а) $$\underbrace{\int_0^{\frac{\pi}{4}} \ln(1+\operatorname{tg} x)\,dx}_{I} = \left[\begin{array}{l} t=\frac{\pi}{4}-x \\ dx=-dt \\ x=0 \to t=\frac{\pi}{4} \\ x=\frac{\pi}{4} \to t=0 \end{array}\right] = -\int_{\frac{\pi}{4}}^{0} \ln\left(1+\operatorname{tg}\left(\frac{\pi}{4}-t\right)\right)dt =$$

применили смо формуле за sin и cos збира и разлике

$$= \int_0^{\frac{\pi}{4}} \ln\left(1+\frac{\sin(\frac{\pi}{4}-t)}{\cos(\frac{\pi}{4}-t)}\right)dt = \int_0^{\frac{\pi}{4}} \ln\left(1+\frac{\frac{\sqrt{2}}{2}\cos t-\frac{\sqrt{2}}{2}\sin t}{\frac{\sqrt{2}}{2}\cos t+\frac{\sqrt{2}}{2}\sin t}\right)dt =$$

$$= \int_0^{\frac{\pi}{4}} \ln\left(\frac{2\cos t}{\cos t+\sin t}\right)dt = \int_0^{\frac{\pi}{4}} \left(\ln 2 - \ln\left(\frac{\cos t+\sin t}{\cos t}\right)\right)dt =$$

$$= t\ln 2\Big|_0^{\frac{\pi}{4}} - \underbrace{\int_0^{\frac{\pi}{4}} \ln(1+\operatorname{tg} t)\,dt}_{I}$$

$$\Rightarrow \quad I = \frac{\pi}{4}\cdot\ln 2 - I \quad \Rightarrow \quad I = \frac{\pi}{8}\cdot\ln 2$$

(б) $$\underbrace{\int_0^1 \frac{\cos^2\frac{\pi x}{2}}{\sqrt{x-x^2}}\,dx}_{I} = \left[\begin{array}{l} t=1-x \\ dx=-dt \\ x=0\to t=1 \\ x=1\to t=0 \end{array}\right] = -\int_1^0 \frac{\cos^2\left(\frac{\pi}{2}-\frac{\pi t}{2}\right)}{\sqrt{1-t-(1-t)^2}}\,dt =$$

корисно
$\cos\left(\frac{\pi}{2}-\alpha\right)=\sin\alpha$

$$= \underbrace{\int_0^1 \frac{\sin^2\left(\frac{\pi t}{2}\right)}{\sqrt{t-t^2}}\,dt}_{I}$$

$$2I = I + I = \int_0^1 \frac{\cos^2\frac{\pi x}{2}}{\sqrt{x-x^2}}\,dx + \int_0^1 \frac{\sin^2\left(\frac{\pi x}{2}\right)}{\sqrt{x-x^2}}\,dx =$$

$$= \int_0^1 \frac{dx}{\sqrt{x-x^2}} = \int_0^1 \frac{dx}{\sqrt{-\left(x-\frac{1}{2}\right)^2+\frac{1}{4}}} = 2\int_0^1 \frac{dx}{\sqrt{1-(2x-1)^2}} =$$

$$= \left[\begin{array}{l} t=2x-1 \\ dx=\frac{1}{2}dt \\ x=0\to t=-1 \\ x=1\to t=1 \end{array}\right] = \int_{-1}^{1} \frac{dt}{\sqrt{1-t^2}} = \arcsin t\Big|_{-1}^{1} = \pi$$

$$\Rightarrow I = \frac{\pi}{2} \quad \blacksquare$$

15. $\displaystyle\int_0^{2\pi} \frac{dx}{(2+\cos x)(3+\cos x)}$

решење

Да је ово неодређен интеграл решавали бисмо га сменом $t = tg\frac{x}{2}$. Овде то не може да се ради јер функција $tg\frac{x}{2}$ није бијекција на $[0,\pi]$! Срећом, то можемо лако да средимо.

1. начин

на $[0,\pi]$ и на $[\pi,2\pi]$ фја $tg\frac{x}{2}$ јесте бијекција па овде може смена

$$\int_0^{2\pi} \frac{dx}{(2+\cos x)(3+\cos x)} = \int_0^{\pi} \frac{dx}{(2+\cos x)(3+\cos x)} + \int_{\pi}^{2\pi} \frac{dx}{(2+\cos x)(3+\cos x)} =$$

смена за оба интеграла:

$t = tg\frac{x}{2}$

$\cos x = \frac{1-t^2}{1+t^2}$

$dx = \frac{2}{1+t^2}dt$

$x=0 \Rightarrow t=0$

$x=\pi \Rightarrow t=\pm\infty$

$x=2\pi \Rightarrow t=0$

$$= \int_0^{+\infty} \frac{t^2+1}{(3+t^2)(2+t^2)}dt + \int_{-\infty}^{0} \frac{t^2+1}{(3+t^2)(2+t^2)}dt =$$

$$= \int_{-\infty}^{+\infty} \frac{t^2+1}{(3+t^2)(2+t^2)}dt = 2\int_{-\infty}^{+\infty}\frac{dt}{3+t^2} - \int_{-\infty}^{+\infty}\frac{dt}{2+t^2} =$$

ово су несвојствени интеграли и текпреба да их радимо

$$= \frac{2}{\sqrt{3}}\operatorname{arctg}\left(\frac{t}{\sqrt{3}}\right)\Bigg|_{-\infty}^{+\infty} - \frac{1}{\sqrt{2}}\operatorname{arctg}\left(\frac{t}{\sqrt{2}}\right)\Bigg|_{-\infty}^{+\infty} =$$

$$= \frac{2}{\sqrt{3}}\cdot\frac{\pi}{2} - \frac{2}{\sqrt{3}}\cdot\left(-\frac{\pi}{2}\right) - \frac{1}{\sqrt{2}}\cdot\frac{\pi}{2} + \frac{1}{\sqrt{2}}\cdot\left(-\frac{\pi}{2}\right) = \frac{2}{\sqrt{3}}\pi - \frac{1}{\sqrt{2}}\pi$$

2. начин

$$\int_0^{2\pi}\frac{dx}{(2+\cos x)(3+\cos x)}=\left[\begin{array}{l} t=x-\pi\\ dt=dx\\ x=0\rightarrow t=-\pi\\ x=2\pi\rightarrow t=\pi\end{array}\right]=\int_{-\pi}^{\pi}\frac{dt}{(2+\cos(\pi+t))(3+\cos(\pi+t))}=$$

$$=\int_{-\pi}^{\pi}\frac{dt}{(2-\cos t)(3-\cos t)}=\left[\begin{array}{l} p=\operatorname{tg}\frac{t}{2}\\ \cos t=\frac{p^2-1}{p^2+1}\\ dt=\frac{2dp}{p^2+1}\\ t=-\pi\rightarrow p=-\infty\\ t=\pi\rightarrow p=+\infty\end{array}\right]=\int_{-\infty}^{+\infty}\frac{p^2+1}{(3+p^2)(2+p^2)}dp=$$

tg $\frac{t}{2}$ јесте бијекција на $(-\pi,\pi)$

$$=\frac{2}{\sqrt3}\pi-\frac{1}{\sqrt2}\pi$$

ПОГРЕШАН начин :

$$\int_0^{2\pi}\frac{dx}{(2+\cos x)(3+\cos x)}=\left[\begin{array}{l} t=\operatorname{tg}\frac{x}{2}\\ \cos x=\frac{t^2-1}{t^2+1}\\ dx=\frac{2}{t^2+1}dt\\ x=0\rightarrow t=0\\ x=2\pi\rightarrow t=0\end{array}\right]=\int_0^0\frac{t^2+1}{(3+t^2)(2+t^2)}dt=0$$

не можемо увести ову смену јер tg $\frac{x}{2}$ није дефинисана у π, па није бијекција на $(0,2\pi)$

16. (а) $\displaystyle\int_0^{2\pi} \frac{dx}{\cos^4 x+\sin^4 x}$; (б) $\displaystyle\int_0^{\pi} \frac{x\sin x}{1+\sin^2 x}\,dx$ (ЈАН1)

решење

(а) <u>I начин</u>:

$$\int_0^{2\pi} \frac{dx}{\cos^4 x+\sin^4 x} = \int_0^{\frac{\pi}{2}} + \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} + \int_{\frac{3\pi}{2}}^{2\pi} =$$

$$\left[\begin{array}{l|l} t=\operatorname{tg}x & x=0 \to t=0 \\ dx=\dfrac{dt}{1+t^2} & x=\frac{\pi}{2} \to t=\pm\infty \\ \sin^2 x=\dfrac{t^2}{1+t^2} & x=\frac{3\pi}{2} \to t=\pm\infty \\ \cos^2 x=\dfrac{1}{1+t^2} & x=2\pi \to t=0 \end{array}\right]$$

овде не може $t=\operatorname{tg}x$ јер $\operatorname{tg}x$ није бијекција на $(0,2\pi)$

на интервалима $(0,\frac{\pi}{2}), (\frac{\pi}{2},\frac{3\pi}{2}), (\frac{3\pi}{2},2\pi)$ ф-ја $\operatorname{tg}x$ је бијекција па може смена $t=\operatorname{tg}x$

рационална функција

$$= \int_0^{+\infty} \frac{t^2+1}{t^4+1}\,dt + \int_{-\infty}^{+\infty} \frac{t^2+1}{t^4+1}\,dt + \int_{-\infty}^{0} \frac{t^2+1}{t^4+1}\,dt = 2\int_{-\infty}^{+\infty} \frac{t^2+1}{t^4+1}\,dt =$$

$$= \int_{-\infty}^{+\infty} \frac{dt}{t^2+t\sqrt{2}+1} + \int_{-\infty}^{+\infty} \frac{dt}{t^2-t\sqrt{2}+1} = \sqrt{2}\operatorname{arctg}(t\sqrt{2}+1)+\sqrt{2}\operatorname{arctg}(t\sqrt{2}-1)\Big|_{-\infty}^{+\infty} =$$

смена $p=t\sqrt{2}+1$

смена $p=t\sqrt{2}-1$

$$= \sqrt{2}\cdot\frac{\pi}{2}+\sqrt{2}\cdot\frac{\pi}{2} - \sqrt{2}\cdot\left(-\frac{\pi}{2}\right) - \sqrt{2}\cdot\left(-\frac{\pi}{2}\right) = 2\sqrt{2}\pi$$

II начин:

$$\int_0^{2\pi}\frac{dx}{\cos^4x+\sin^4x}=\int_0^{\pi}+\int_{\pi}^{2\pi}=\left[\begin{array}{l|l} t=tg\frac{x}{2} & x=0\to t=0\\ dx=\frac{2dt}{1+t^2} & x=\pi\to t=\pm\infty\\ \sin x=\frac{2t}{1+t^2} & x=2\pi\to t=0\\ \cos x=\frac{1-t^2}{1+t^2} & \end{array}\right]=$$

$$=\int_0^{+\infty}\frac{2(1+t^2)^3}{(1-t^2)^4+16t^4}dt+\int_{-\infty}^{0}\frac{2(1+t^2)^3}{(1-t^2)^4+16t^4}dt=$$

$$=\int_{-\infty}^{+\infty}\frac{2(1+t^2)^3}{(1-t^2)^4+16t^4}dt=\ldots$$

III начин:

Приметимо: $\dfrac{1}{\cos^4(x+\frac{\pi}{2})+\sin^4(x+\frac{\pi}{2})}=\dfrac{1}{\sin^4x+\cos^4x}$

тј. период подинтегралне функције је $\frac{\pi}{2}$.

$$\int_0^{2\pi}\frac{dx}{\cos^4x+\sin^4x}=4\int_0^{\frac{\pi}{2}}\frac{dx}{\cos^4x+\sin^4x}=4\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\frac{dx}{\cos^4x+\sin^4x}=$$

$\boxed{\text{парна}}$

$$=8\int_0^{\frac{\pi}{4}}\frac{dx}{\cos^4x+\sin^4x}=\left[\begin{array}{l|l} t=tg\,x & x=0\to t=0\\ dx=\frac{dt}{1+t^2} & x=\frac{\pi}{4}\to t=1\\ \cos^2x=\frac{1}{1+t^2} & \\ \sin^2x=\frac{t^2}{1+t^2} & \end{array}\right]=$$

$$= 8\int_0^1 \frac{t^2+1}{t^4+1}\,dt = 8\sqrt{2}\,\mathrm{arctg}(t\sqrt{2}+1) + 8\sqrt{2}\,\mathrm{arctg}(t\sqrt{2}-1)\Big|_0^1 =$$

исти интеграл из 1. начина

$$= 8\sqrt{2}\,\mathrm{arctg}(\sqrt{2}+1) + 8\sqrt{2}\,\mathrm{arctg}(\sqrt{2}-1) - 8\sqrt{2}\cdot\frac{\pi}{4} - 8\sqrt{2}\cdot\left(-\frac{\pi}{4}\right)$$

$$= 8\sqrt{2}\,\mathrm{arctg}(\sqrt{2}+1) + 8\sqrt{2}\,\mathrm{arctg}(\sqrt{2}-1)$$

ПОГРЕШАН начин:

$$\int_0^{2\pi} \frac{dx}{\cos^4 x+\sin^4 x} = \begin{bmatrix} t=\mathrm{tg}\,x \\ \dots \\ x=0 \to t=0 \\ x=2\pi \to t=0 \end{bmatrix} = \int_0^0 \frac{t^2+1}{t^4+1}\,dt = 0.$$

(б) $$\int_0^{\pi} \frac{x\sin x}{1+\sin^2 x}\,dx = \begin{bmatrix} t=x-\frac{\pi}{2} \\ dx=dt \\ x=0 \to t=-\frac{\pi}{2} \\ x=\pi \to t=\frac{\pi}{2} \end{bmatrix} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{(t+\frac{\pi}{2})\sin(t+\frac{\pi}{2})}{1+\sin^2(t+\frac{\pi}{2})}\,dt =$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{(t+\frac{\pi}{2})\cos t}{1+\cos^2 t}\,dt = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{t\cos t}{1+\cos^2 t}\,dt + \frac{\pi}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cos t}{1+\cos^2 t}\,dt =$$

Непарна ПАРНА

$$= \pi\int_0^{\frac{\pi}{2}} \frac{\cos t}{2-\sin^2 t}\,dt = \begin{bmatrix} p=\sin t \\ dp=\cos t\,dt \\ t=0 \to p=0 \\ t=\frac{\pi}{2} \to p=1 \end{bmatrix} = \pi\int_0^1 \frac{dp}{2-p^2} =$$

$$= \frac{\pi}{2\sqrt{2}} \ln(p+\sqrt{2}) - \frac{\pi}{2\sqrt{2}} \ln(\sqrt{2}-p) \Big|_0^1 =$$

$$= \frac{\pi}{2\sqrt{2}} \ln(1+\sqrt{2}) - \frac{\pi}{2\sqrt{2}} \ln(\sqrt{2}-1) - \frac{\pi}{2\sqrt{2}} \ln(\sqrt{2}) + \frac{\pi}{2\sqrt{2}} \ln(\sqrt{2}) =$$

$$= \frac{\pi}{2\sqrt{2}} \ln\left(\frac{1+\sqrt{2}}{\sqrt{2}-1}\right) \quad \blacksquare$$

$\boxed{17.}$ Нека је $I_n = \int_0^1 x^n e^{\alpha x}\,dx$, $n \in \mathbb{N}_0$, $\alpha \in \mathbb{R} \setminus \{0\}$.

(а) Наћи везу између I_n и I_{n-1}, $n \geq 1$;

(б) Израчунати I_3.

Решење

$$\text{(а)}\quad I_n = \int_0^1 x^n e^{\alpha x}\,dx = \begin{bmatrix} u = x^n & dv = e^{\alpha x}dx \\ du = nx^{n-1}dx & v = \frac{1}{\alpha} e^{\alpha x} \end{bmatrix} =$$

$$= \frac{1}{\alpha} x^n \cdot e^{\alpha x} \Big|_0^1 - \frac{n}{\alpha} \int_0^1 x^{n-1} e^{\alpha x}\,dx =$$

$$= \frac{e^{\alpha}}{\alpha} - \frac{n}{\alpha} I_{n-1}$$

$$\text{(б)}\quad I_3 = \frac{e^{\alpha}}{\alpha} - \frac{3}{\alpha} I_2 = \frac{e^{\alpha}}{\alpha} - \frac{3}{\alpha}\left(\frac{e^{\alpha}}{\alpha} - \frac{2}{\alpha} I_1\right) =$$

$$= \frac{e^{\alpha}}{\alpha} - \frac{3}{\alpha}\left(\frac{e^{\alpha}}{\alpha} - \frac{2}{\alpha}\left(\frac{e^{\alpha}}{\alpha} - \frac{1}{\alpha} I_0\right)\right) =$$

$$= \frac{e^\alpha}{\alpha} - \frac{3e^\alpha}{\alpha^2} + \frac{6e^\alpha}{\alpha^3} - \frac{6}{\alpha^3} I_0 =$$

$$= e^\alpha \cdot \frac{\alpha^2 - 3\alpha + 6}{\alpha^3} - \frac{6}{\alpha^3} \cdot \int_0^1 e^{\alpha x} dx =$$

$$= e^\alpha \frac{\alpha^2 - 3\alpha + 6}{\alpha^3} - \left(\frac{6}{\alpha^4} e^{\alpha x} \right) \Big|_0^1 =$$

$$= e^\alpha \frac{\alpha^2 - 3\alpha + 6}{\alpha^3} - \frac{6e^\alpha}{\alpha^4} + \frac{6}{\alpha^4} =$$

$$= e^\alpha \frac{\alpha^3 - 3\alpha^2 + 6\alpha - 6}{\alpha^4} + \frac{6}{\alpha^4} \quad \blacksquare$$