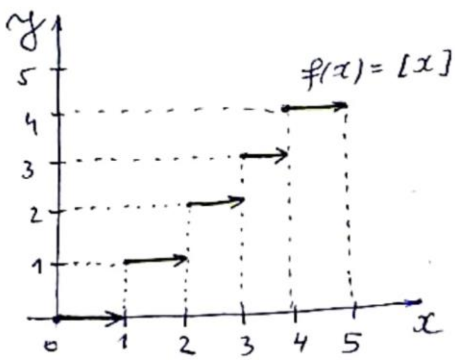
$\boxed{2.}$ (а) $\int_{\frac{1}{2}}^{3} [x]\,dx$; (б) $\int_{0}^{2} [e^x]\,dx$; (в) $\int_{1}^{n+1} \ln [x]\,dx$

решење

(а) $f(x) = [x]$ ← цео део (тј. $[x] = \text{floor}(x)$)



нпр. $[2,1] = 2$

$[\sqrt{2}] = 1$

$[3] = 3$

$$\int_{\frac{1}{2}}^{3} [x]\,dx = \int_{\frac{1}{2}}^{1} \underbrace{[x]}_{0}\,dx + \int_{1}^{2} \underbrace{[x]}_{1}\,dx + \int_{2}^{3} \underbrace{[x]}_{2}\,dx =$$

$$= \int_{1}^{2} dx + \int_{2}^{3} 2\,dx = x\Big|_{1}^{2} + 2x\Big|_{2}^{3} =$$

$$= 2 - 1 + 6 - 4 = 3$$

(б) $[e^x] = m \Rightarrow m \leq e^x < m+1$

$$\ln m \leq x \leq \ln(m+1)$$

$$\int_{0}^{2} [e^x]\,dx = \int_{\ln 1}^{\ln 2} [e^x]\,dx + \int_{\ln 2}^{\ln 3} [e^x]\,dx + \dots + \int_{\ln 7}^{2} [e^x]\,dx - Ⓐ$$

имамо $2 = \ln e^2 \approx \ln 7{,}4$ па је зато последњи интеграл од $\ln 7$ до 2.

$$A = \int_{\ln 1}^{\ln 2} dx + 2\int_{\ln 2}^{\ln 3} dx + 3\int_{\ln 3}^{\ln 4} dx + 4\int_{\ln 4}^{\ln 5} dx + 5\int_{\ln 5}^{\ln 6} dx + 6\int_{\ln 6}^{\ln 7} dx + 7\int_{\ln 7}^{2} dx =$$

$$= x\Big|_{\ln 1}^{\ln 2} + 2x\Big|_{\ln 2}^{\ln 3} + 3x\Big|_{\ln 3}^{\ln 4} + 4x\Big|_{\ln 4}^{\ln 5} + 5x\Big|_{\ln 5}^{\ln 6} + 6x\Big|_{\ln 6}^{\ln 7} + 7x\Big|_{\ln 7}^{2} =$$

$$= \ln 2 - \ln 1 + 2\ln 3 - 2\ln 2 + 3\ln 4 - 3\ln 3 + 4\ln 5 - 4\ln 4 + 5\ln 6 - 5\ln 5 +$$

$$+ 6\ln 7 - 6\ln 6 + 14 - 7\ln 7 =$$

$$= -\ln 2 - \ln 3 - \ln 4 - \ln 5 - \ln 6 - \ln 7 + 14 = 14 - \ln(7!)$$

(в) $$\int_1^{n+1} \ln[x]\,dx = \int_1^2 \ln 1\,dx + \int_2^3 \ln 2\,dx + \int_3^4 \ln 3\,dx + \dots + \int_n^{n+1} \ln n\,dx =$$

$$= x\ln 2\Big|_2^3 + x\ln 3\Big|_3^4 + \dots + x\ln n\Big|_n^{n+1} = \ln 2 + \ln 3 + \dots + \ln n = \ln(n!)$$ ∎

3. (a) $\int_0^1 x\ln(1+x^2)\,dx$; (б) $\int_0^{\sqrt{3}} x\,\mathrm{arctg}\,x\,dx$

решение

(a) $$\int_0^1 x\ln(1+x^2)\,dx = \begin{bmatrix} u = \ln(1+x^2) & dv = x\,dx \\ du = \frac{2x}{1+x^2}\,dx & v = \frac{1}{2}x^2 \end{bmatrix} =$$

$$= \frac{1}{2}x^2\ln(1+x^2)\Big|_0^1 - \int_0^1 \frac{x^2\cdot x}{1+x^2}\,dx = \begin{bmatrix} t = 1+x^2 \\ dt = 2x\,dx \\ x=0 \to t=1 \\ x=1 \to t=2 \end{bmatrix} =$$

$$= \frac{\ln 2}{2} - \frac{1}{2}\int_1^2 \frac{t-1}{t}\,dt = \frac{\ln 2}{2} - \left(\frac{1}{2}t - \frac{1}{2}\ln|t|\right)\Big|_1^2 =$$

$$= \frac{\ln 2}{2} - \left(\frac{1}{2}\cdot 2 - \frac{1}{2}\ln 2 - \frac{1}{2}\cdot 1 + \frac{1}{2}\ln 1 \right) =$$

$$= \ln 2 - \frac{1}{2}$$

(б) $$\int_0^{\sqrt{3}} x \operatorname{arctg} x \, dx = \begin{bmatrix} u = \operatorname{arctg} x & dv = x\,dx \\ du = \dfrac{dx}{1+x^2} & v = \frac{1}{2}x^2 \end{bmatrix} =$$

$$= \frac{1}{2}x^2 \operatorname{arctg} x \Big|_0^{\sqrt{3}} - \frac{1}{2}\int_0^{\sqrt{3}} \frac{x^2+1-1}{1+x^2}\, dx =$$

$$= \frac{3}{2}\cdot\frac{\pi}{3} - \frac{1}{2}\int_0^{\sqrt{3}} \left(1 - \frac{1}{1+x^2}\right) dx =$$

$$= \frac{\pi}{2} - \left(\frac{1}{2}x - \frac{1}{2}\operatorname{arctg} x\right)\Big|_0^{\sqrt{3}} =$$

$$= \frac{\pi}{2} - \left(\frac{\sqrt{3}}{2} - \frac{\pi}{6}\right) = \frac{2\pi}{3} - \frac{\sqrt{3}}{2} \quad \blacksquare$$

4. По дефиницији израчунати

(а) $\int_0^1 e^x dx$; (б) $\int_1^2 \frac{1}{x^2} dx$

решење

(а) $[0,1]$ поделимо на n делова:

$$0 = x_0 \quad x_1 \quad x_2 \quad \cdots \quad x_{n-1} \quad x_n = 1 \qquad x_i = \frac{i}{n}$$

Узмимо $\xi_i = x_i$ ($\xi_i \in [x_{i-1}, x_i]$ бирамо произвољно)

$$\sigma(f, [0,1], P, \xi) = \sum_{i=1}^{n} (x_i - x_{i-1}) \cdot f(\xi_i) =$$

$$= \sum_{i=1}^{n} \underbrace{\left(\frac{i}{n} - \frac{i-1}{n}\right)}_{\frac{1}{n}} \cdot e^{\xi_i} = \frac{1}{n} \sum_{i=1}^{n} e^{\frac{i}{n}} =$$

$$= \frac{1}{n} \left(e^{\frac{1}{n}} + \left(e^{\frac{1}{n}}\right)^2 + \left(e^{\frac{1}{n}}\right)^3 + \dots + \left(e^{\frac{1}{n}}\right)^n \right) =$$

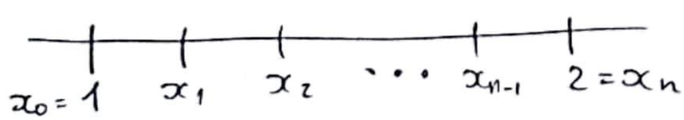
$$= \frac{1}{n} \cdot e^{\frac{1}{n}} \cdot \frac{1 - \left(e^{\frac{1}{n}}\right)^n}{1 - e^{\frac{1}{n}}} = \frac{1}{n} e^{\frac{1}{n}} \cdot \frac{1-e}{1-e^{\frac{1}{n}}} =$$

$$= e^{\frac{1}{n}} (e-1) \cdot \frac{\frac{1}{n}}{1 - e^{\frac{1}{n}}}$$

$$\int_0^1 e^x dx = \lim_{n\to\infty} \sigma(f, [0,1], P, \xi) = \lim_{n\to\infty} e^{\frac{1}{n}} (e-1) \frac{\frac{1}{n}}{e^{\frac{1}{n}} - 1} = e-1$$

$$\boxed{\text{Коришћено: } \lim_{n\to\infty} \sqrt[n]{e} = 1 \qquad \lim_{n\to\infty} \frac{e^{\frac{1}{n}} - 1}{\frac{1}{n}} = 1}$$

(б) $[1,2]$ делимо на n делова:



$$x_i = 1 + \frac{i}{n}$$

$$\Delta x_i = x_i - x_{i-1} = \frac{1}{n}$$

$\xi_i = \sqrt{x_i x_{i-1}}$ (овде за ξ бирамо корен јер је $f(x) = \frac{1}{x^2}$)

$$\sigma(f,[1,2],P,\xi)=\sum_{i=1}^{n}(x_i-x_{i-1})\cdot f(\xi_i)=$$

$$=\sum_{i=1}^{n}\frac{1}{n}\cdot\frac{1}{(\sqrt{x_i x_{i-1}})^2}=\frac{1}{n}\sum_{i=1}^{n}\frac{1}{x_i x_{i-1}}=\frac{1}{n}\sum_{i=1}^{n}\frac{1}{(1+\frac{i}{n})(1+\frac{i-1}{n})}=$$

$$=\frac{1}{n}\left(\frac{1}{(1+\frac{1}{n})(1+\frac{0}{n})}+\frac{1}{(1+\frac{2}{n})(1+\frac{1}{n})}+\dots+\frac{1}{(1+\frac{n}{n})(1+\frac{n-1}{n})}\right)=$$

$$=\frac{1}{n}\left(\frac{n}{1+\frac{0}{n}}-\frac{n}{1+\frac{1}{n}}+\frac{n}{1+\frac{1}{n}}-\frac{n}{1+\frac{2}{n}}+\dots+\frac{n}{1+\frac{n-1}{n}}-\frac{n}{1+\frac{n}{n}}\right)=$$

$$=\frac{1}{n}\left(\frac{n}{1+\frac{0}{n}}-\frac{n}{1+\frac{n}{n}}\right)=\frac{1}{n}\left(n-\frac{n}{2}\right)=\frac{1}{n}\cdot\frac{n}{2}=\frac{1}{2}$$

$$\int_1^2\frac{1}{x^2}\,dx\overset{def}{=}\lim_{n\to\infty}\sigma(f,[1,2],P,\xi)=\lim_{n\to\infty}\frac{1}{2}=\frac{1}{2}\quad\blacksquare$$

$\boxed{5.}$ Наћи извод функције $F(x)$ ако је

(а) $F(x)=\int_0^x e^{-t^2}dt$; (б) $F(x)=\int_x^0\sqrt{1+t^3}\,dt$;

(в) $F(x)=\int_{\ln x}^{2\ln x}\frac{e^t}{t}\,dt$

решење

(а) $F'(x)=\left(\int_0^x e^{-t^2}dt\right)'=e^{-x^2}$

(б) $F'(x) = \left(\int_x^0 \sqrt{1+t^3}\, dt \right)' = -\left(\int_0^x \sqrt{1+t^3}\, dt \right)' = -\sqrt{1+x^3}$

(в) $\int_{\ln x}^{2\ln x} \frac{e^t}{t}\, dt = \left[\begin{array}{l} p = e^t \\ dt = \frac{dp}{p} \\ t = \ln x \to p = x \\ t = 2\ln x \to p = x^2 \end{array} \right] = \int_x^{x^2} \frac{p}{\ln p} \cdot \frac{1}{p}\, dp =$

$= \int_x^{x^2} \frac{dp}{\ln p} = \int_x^0 \frac{dp}{\ln p} + \int_0^{x^2} \frac{dp}{\ln p} = \left[\begin{array}{l} \text{смена за 2. инт} \\ y^2 = p \\ dp = 2y\,dy \\ p = 0 \to y = 0 \\ p = x^2 \to y = x \end{array} \right] =$

$= -\int_0^x \frac{dp}{\ln p} + \int_0^x \frac{2y\,dy}{\ln y^2}$

$$F'(x) = \left(-\int_0^x \frac{dp}{\ln p} + \int_0^x \frac{2y\,dy}{\ln y^2} \right)' = -\frac{1}{\ln x} + \frac{2x}{\ln x^2} = -\frac{1}{\ln x} + \frac{2x}{2\ln x} = \frac{x-1}{\ln x}$$

други начин коришћењем формуле

$$\left(\int_{a(x)}^{b(x)} f(t)\, dt \right)' = f(b(x)) \cdot b'(x) - f(a(x)) \cdot a'(x)$$

$$\left(\int_{\ln x}^{2\ln x} \frac{e^t}{t}\, dt \right)' = \frac{e^{2\ln x}}{2\ln x} \cdot \frac{2}{x} - \frac{e^{\ln x}}{\ln x} \cdot \frac{1}{x} = \frac{x-1}{\ln x} \quad \square$$

6 Коришћењем Лопиталовог правила наћи лимесе:

(а) $\lim_{x \to +\infty} \frac{1}{x^2} \int_0^x \sqrt{1+t^4}\, dt$; (б) $\lim_{x \to 0+} \frac{1}{x^3} \int_0^{x^3} \sin\sqrt{t}\, dt$

решење

(а) $\lim_{x \to +\infty} \frac{\int_0^x \sqrt{1+t^4}\, dt}{x^2} \overset{\text{Л.П.}}{\underset{\text{„}\frac{\infty}{\infty}\text{"}}{=}} \lim_{x \to +\infty} \frac{\left(\int_0^x \sqrt{1+t^4}\, dt \right)'}{2x} = \lim_{x \to +\infty} \frac{\sqrt{1+x^4}}{x} = +\infty$

(б) $\int_0^{x^3} \sin\sqrt{t}\,dt = \left[\begin{array}{l} p^3 = t \\ dt = 3p^2dp \\ t=0 \to p=0 \\ t=x^3 \to p=x \end{array}\right] = \int_0^{x} \sin\sqrt{p^3} \cdot 3p^2 dp$

$$\lim_{x\to0+} \frac{\int_0^{x^3} \sin\sqrt{t}\,dt}{x^3} = \lim_{x\to0+} \frac{\int_0^{x} \sin\sqrt{p^3} \cdot 3p^2dp}{x^3} \overset{\text{Л.П.}}{\underset{„\frac{0}{0}“}{=}}$$

$$= \lim_{x\to0+} \frac{\left(\int_0^{x} \sin\sqrt{p^3} \cdot 3p^2dp\right)'}{3x^2} = \lim_{x\to0+} \frac{\sin\sqrt{x^3} \cdot 3x^2}{3x^2} =$$

$$= \lim_{x\to0+} \sin\sqrt{x^3} = 0 \quad \blacksquare$$

$\boxed{7.}$ Израчунати коришћењем интегралних сума:

(а) $\lim\limits_{n\to\infty} \frac{1}{n}\left(\sin\frac{\pi}{n} + \sin\frac{2\pi}{n} + \dots + \sin\frac{(n-1)\pi}{n}\right)$

(б) $\lim\limits_{n\to\infty} \frac{1^p+2^p+\dots+n^p}{n^{p+1}}$, $p>1$

решење

(а) $\lim\limits_{n\to\infty} \frac{1}{n}\left(\sin\frac{\pi}{n} + \sin\frac{2\pi}{n} + \dots + \sin\frac{(n-1)\pi}{n}\right) =$

$$= \lim_{n\to\infty} \frac{\pi}{n}\cdot\frac{1}{\pi}\Big(\underbrace{\sin\frac{0\pi}{n}}_{0} + \sin\frac{\pi}{n} + \dots + \frac{\sin(n-1)\pi}{n} + \underbrace{\sin\frac{n\pi}{n}}_{0}\Big) =$$

$$= \frac{1}{\pi}\int_0^{\pi} \sin x\,dx = -\frac{1}{\pi}\cos x \Big|_0^{\pi} = \frac{-\cos\pi + \cos 0}{\pi} = \frac{2}{\pi}$$

(б) $$\lim_{n\to\infty} \frac{1^p+2^p+\dots+n^p}{n^{p+1}} = \lim_{n\to\infty} \frac{1}{n}\left(\left(\frac{1}{n}\right)^p + \left(\frac{2}{n}\right)^p + \dots + \left(\frac{n}{n}\right)^p\right) =$$

$$= \int_0^1 x^p\,dx = \frac{1}{p+1}\,x^{p+1}\Big|_0^1 = \frac{1}{p+1}$$ ■

задатке $\boxed{8.}$ и $\boxed{9.}$ на испиту користимо без доказа!

$\boxed{8.}$ Нека је f интеграбилна.

Доказати:

(а) ако је f парна, тј. $(\forall x)\ f(-x)=f(x)$,

онда $\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$;

(б) ако је f непарна, тј. $(\forall x)\ f(-x)=-f(x)$,

онда $\int_{-a}^{a} f(x)\,dx = 0$.

решење

(а) $$\int_{-a}^{a} f(x)dx = \int_{-a}^{0} f(x)dx + \int_0^a f(x)dx = \left[\begin{array}{l}\text{смена за 1. интеграл:}\\ t=-x\\ dx=-dt\\ x=-a \to t=a\\ x=0 \to t=0\end{array}\right] =$$

$$= -\int_a^0 f(-t)\,dt + \int_0^a f(x)\,dx = \int_0^a f(t)\,dt + \int_0^a f(x)\,dx = 2\int_0^a f(x)\,dx$$

(б) $$\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_0^a f(x)\,dx = \left[\begin{array}{l} \text{смена за} \\ \text{1. инт.} \\ x=-t \\ dx=-dt \\ x=-a \to t=a \\ x=0 \to t=0 \end{array}\right] =$$

$$= -\int_a^0 f(-t)\,dt + \int_0^a f(x)\,dx =$$

$$= -\int_0^a f(t)\,dt + \int_0^a f(x)\,dx = 0 \quad \blacksquare$$

9. Нека је f интеграбилна и периодична са периодом T (тј. $(\forall x)\ f(x+T)=f(x)$). Доказати

(а) $\displaystyle\int_a^{a+T} f(x)\,dx = \int_0^T f(x)\,dx, \quad a\in\mathbb{R};$

(б) $\displaystyle\int_a^{a+kT} f(x)\,dx = k\cdot\int_0^T f(x)\,dx, \quad a\in\mathbb{R},\ k\in\mathbb{Z};$

(в) $\displaystyle\int_{a+kT}^{b+kT} f(x)\,dx = \int_a^b f(x)\,dx, \quad a,b\in\mathbb{R}.$

решење

(а) $$\int_a^{a+T} f(x)\,dx = \int_a^T f(x)\,dx + \int_T^{a+T} f(x)\,dx = \int_a^T f(x)\,dx + \int_T^{a+T} f(x-T)\,dx =$$

$$= \left[\begin{array}{l} \text{смена за} \\ \text{2. инт.} \\ t=x-T \\ dt=dx \\ x=T \to t=0 \\ x=a+T \to t=a \end{array}\right] = \int_a^T f(x)\,dx + \int_0^a f(t)\,dt = \int_0^T f(x)\,dx$$

б) $\int_a^{a+kT} f(x)\,dx = \int_a^{a+T} f(x)dx + \int_{a+T}^{a+2T} f(x)dx + \dots + \int_{a+(k-1)T}^{a+kT} f(x)\,dx =$

$$\overset{(a)}{=} \underbrace{\int_0^T f(x)dx + \int_0^T f(x)\,dx + \dots + \int_0^T f(x)\,dx}_{k} =$$

$$= k \int_0^T f(x)dx$$

(в) $\int_{a+kT}^{b+kT} f(x)dx = \int_{a+kT}^{b+kT} f(x-kT)\,dx = \begin{bmatrix} t = x - kT \\ dt = dx \\ x = a+kT \to t = a \\ x = b+kT \to t = b \end{bmatrix} = \int_a^b f(t)dt$ ■

Претходна два задатка су јако битна !