$\boxed{25.}$ (а) $\displaystyle\int \frac{2x^2-1}{x\sqrt{1-x^2}}\,dx$ (б) $\displaystyle\int \frac{\sqrt{1-x^2}}{x^2}\,dx$ (ЈУН1)

решење

(а) $$\int \frac{2x^2-1}{x\sqrt{1-x^2}}\,dx = \begin{bmatrix} x = \sin t \\ dx = \cos t\,dt \end{bmatrix} = \int \frac{2\sin^2 t - 1}{\sin t\sqrt{\underbrace{1-\sin^2 t}_{\cos^2 t}}}\cdot \cos t\,dt =$$

случај (б)

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$$= \int \frac{2\sin^2 t-1}{\sin t}\,dt = 2\int \sin t\,dt - \int \frac{dt}{\sin t} =$$

$$= -2\cos t - \frac{1}{2}\ln\left|\frac{\cos t-1}{\cos t+1}\right| + c =$$

$$= -2\cos(\arcsin x) - \frac{1}{2}\ln\left|\frac{\cos(\arcsin x)-1}{\cos(\arcsin x)+1}\right| + c =$$

$$= -2\sqrt{1-x^2} - \frac{1}{2}\ln\left|\frac{\sqrt{1-x^2}-1}{\sqrt{1-x^2}+1}\right| + c$$

Користили смо
$\cos(\arcsin x) = \sqrt{1-x^2}$

(б) $$\int \frac{\sqrt{1-x^2}}{x^2}\,dx = \begin{bmatrix} x = \sin t \\ dx = \cos t\,dt \end{bmatrix} = \int \frac{\sqrt{1-\sin^2 t}}{\sin^2 t}\cos t\,dt = \int \frac{\cos^2 t}{\sin^2 t}\,dt =$$

$$= \int \frac{1-\sin^2 t}{\sin^2 t}\,dt = \int\left(\frac{1}{\sin^2 t} - 1\right)dt = -\operatorname{ctg} t - t + C =$$

$$= -\operatorname{ctg}(\arcsin x) - \arcsin x + C. \quad \blacksquare$$

Интеграли (а) и (б) на стр. 32 се могу решавати и овим сменама:

(а) $\int R_1(\sqrt{x^2+1}, x)\,dx$ $\boxed{\text{смена} \quad x = sh\, t}$

(б) $\int R_1(\sqrt{x^2-1}, x)\,dx$ $\boxed{\text{смена} \quad x = ch\, t}$

Хиперболичке функције:

$$sh\, x \overset{def}{=} \frac{e^x - e^{-x}}{2}$$

синус хиперболички

$$ch\, x \overset{def}{=} \frac{e^x + e^{-x}}{2}$$

косинус хиперболички

важи:

- $1 + sh^2 x = ch^2 x$
- $(sh\, x)' = ch\, x$
- $(ch\, x)' = sh\, x$
- $\int sh\, x\, dx = ch\, x + c$
- $\int ch\, x\, dx = sh\, x + c$
- $arcsh\, x = \ln(x + \sqrt{x^2+1})$
- $arcch\, x = \ln(x + \sqrt{x^2-1})$

$\boxed{26}$. (а) $\int \sqrt{x^2+1}\,dx$; (б) $\int \frac{dx}{(x+1)\sqrt{x^2-1}}$

решење

$$(а) \int \sqrt{x^2+1}\,dx = \begin{bmatrix} x = sh\, t \\ dx = ch\, t\, dt \end{bmatrix} = \int \sqrt{sh^2 t + 1} \cdot ch\, t\, dt =$$

$$= \int ch^2 t\, dt = \int \left(\frac{e^t + e^{-t}}{2}\right)^2 dt = \frac{1}{4} \int \left(e^{2t} + 2 + e^{-2t}\right) dt =$$

$$= \frac{1}{8} e^{2t} + \frac{1}{2} t - \frac{1}{8} e^{-2t} + c = \frac{1}{8} e^{2\, arcsh\, x} + \frac{1}{2} arcsh\, x - \frac{1}{8} e^{-2\, arcsh\, x} + c =$$

$$= \frac{1}{8}\left(x+\sqrt{x^2+1}\right)^2 + \frac{1}{2}\ln\left(x+\sqrt{x^2+1}\right) - \frac{1}{8}\cdot\frac{1}{\left(x+\sqrt{1+x^2}\right)^2} + c$$

$\boxed{\text{Користимо } e^{\ln x} = x}$

(б) $$\int \frac{dx}{(x+1)\sqrt{x^2-1}} = \begin{bmatrix} x = \operatorname{ch} t \\ dx = \operatorname{sh} t\, dt \end{bmatrix} = \int \frac{\operatorname{sh} t}{(\operatorname{ch} t+1)\sqrt{\operatorname{ch}^2 t-1}}\, dt =$$

$$= \int \frac{dt}{\operatorname{ch} t+1} = \int \frac{dt}{\frac{e^t+e^{-t}}{2}+1} = 2\int \frac{dt}{e^t+e^{-t}+2} \begin{matrix} \cdot e^t \\ = \\ \cdot e^t \end{matrix}$$

$$= 2\int \frac{e^t}{e^{2t}+2e^t+1}\, dt = 2\int \frac{e^t}{(e^t+1)^2}\, dt = \begin{bmatrix} e^t+1 = p \\ e^t dt = dp \end{bmatrix} =$$

$$= 2\int \frac{dp}{p^2} = -\frac{2}{p} + c = -\frac{2}{e^t+1} + c =$$

$$= -\frac{2}{e^{\operatorname{arcch} x}+1} + c = -\frac{2}{x+1+\sqrt{x^2-1}} + c \quad \blacksquare$$

$\boxed{\text{Користимо } e^{\ln x} = x}$

За крај интеграла овог типа упознаћемо се са Ојлеровим сменама.

Ојлерове смене

решавају интеграле облика $\int R\left(\sqrt{ax^2+bx+c},\, x\right)dx$

исти облик са стр. 32!

① $a>0$: смена $\sqrt{ax^2+bx+c} = t \pm \sqrt{a}\, x$

② $c>0$: смена $\sqrt{ax^2+bx+c} = xt \pm \sqrt{c}$

+ или − бирамо како нам је згодно

③ $ax^2+bx+c = a(x-x_1)(x-x_2)$

смена $\sqrt{ax^2+bx+c} = t(x-x_1)$

27. $\displaystyle\int \frac{dx}{x+\sqrt{x^2+x+1}}$

решење

$a=1>0$ радимо ① смену (а могла је и ②)

$$\int \frac{dx}{x+\sqrt{x^2+x+1}} = \left[\begin{array}{l} \sqrt{x^2+x+1} = t-x \;/^2 \\ x^2+x+1 = t^2-2tx+x^2 \\ x = \dfrac{t^2-1}{1+2t} \\ dx = \dfrac{2t^2+2t+2}{(1+2t)^2}\,dt \end{array}\right] = \int \frac{1}{t}\cdot\frac{2(t^2+t+1)}{(1+2t)^2}\,dt$$

$$= 2\int \frac{t^2+t+1}{t(1+2t)^2}\,dt = \boxed{\begin{array}{c}\text{интеграција}\\ \text{рационалне}\\ \text{функције}\\ \dots\end{array}} = 2\ln|t| - \frac{3}{2}\ln|1+2t| + \frac{3}{1+2t} + C$$

$$= 2\ln\left|x+\sqrt{x^2+x+1}\right| - \frac{3}{2}\ln\left|1+2\left(x+\sqrt{x^2+x+1}\right)\right| + \frac{3}{1+2\left(x+\sqrt{x^2+x+1}\right)} + C$$

$\boxed{28.}$ $\int \frac{dx}{1+\sqrt{1-2x-x^2}}$

решење

$c=1>0$ радимо ② смену (могла је и ③)

$$\int \frac{dx}{1+\sqrt{1-2x-x^2}} = \left[\begin{array}{l} \sqrt{1-2x-x^2} = xt-1 \,/^2 \\ 1-2x-x^2 = x^2t^2-2tx+1 \\ x = \frac{2t-2}{t^2+1} \\ dx = \frac{-2t^2+4t+2}{(t^2+1)^2}\,dt \end{array}\right] =$$

$$= \int \frac{\frac{-2t^2+4t+2}{(t^2+1)^2}}{\frac{2t-2}{t^2+1}\cdot t}\,dt = \int \frac{-t^2+2t+1}{t(t-1)(t^2+1)}\,dt = \boxed{\text{интеграција рационалне функције} \ldots} =$$

$$= -\ln|t| + \ln|t-1| - 2\operatorname{arctg} t + c =$$

$$= -\ln\left|\frac{1+\sqrt{1-2x-x^2}}{x}\right| + \ln\left|\frac{1+\sqrt{1-2x-x^2}}{x}-1\right| -$$

$$-2\operatorname{arctg}\left(\frac{1+\sqrt{1-2x-x^2}}{x}\right) + c \quad \blacksquare$$

29. $\displaystyle\int \frac{x-\sqrt{x^2+3x+2}}{x+\sqrt{x^2+3x+2}}\,dx$

решење

$x^2+3x+2=(x+1)(x+2)$ радимо 3 смену

(а могли смо и 1 и 2)

$$\int \frac{x-\sqrt{x^2+3x+2}}{x+\sqrt{x^2+3x+2}}\,dx = \left[\begin{array}{l} \sqrt{x^2+3x+2}=t(x+1)\ /^2 \\ (x+1)(x+2)=t^2(x+1)^2 \\ x=\dfrac{t^2-2}{1-t^2} \\ dx=\dfrac{-2t}{(1-t^2)^2}\,dt \end{array}\right] =$$

$$= \int \frac{\dfrac{t^2-2}{1-t^2}-t\left(\dfrac{t^2-2}{1-t^2}+1\right)}{\dfrac{t^2-2}{1-t^2}+t\left(\dfrac{t^2-2}{1-t^2}+1\right)}\cdot\frac{-2t}{(1-t^2)^2}\,dt =$$

$$= \int \frac{t^2+t-2}{t^2-t-2}\cdot\frac{-2t}{(1-t^2)^2}\,dt = \dots \qquad \blacksquare$$

Интеграли облика $\int R\left(x, \sqrt[n]{\frac{ax+b}{cx+d}}\right)dx$

уводимо смену $t^n = \frac{ax+b}{cx+d}$

$\boxed{30.}$ (а) $\int \frac{\sqrt{x+1}-\sqrt{x-1}}{\sqrt{x+1}+\sqrt{x-1}}dx$; (б) $\int \frac{\sqrt{x-2}+1}{\sqrt{x-2}-1}dx$

решење

$$(a) \int \frac{\sqrt{x+1}-\sqrt{x-1}}{\sqrt{x+1}+\sqrt{x-1}}dx \begin{matrix} :\sqrt{x-1} \\ :\sqrt{x-1}\end{matrix} = \int \frac{\sqrt{\frac{x+1}{x-1}}-1}{\sqrt{\frac{x+1}{x-1}}+1}dx = \left[\begin{array}{l} t^2 = \frac{x+1}{x-1} \\ t^2(x-1) = x+1 \\ x = \frac{t^2+1}{t^2-1} \\ dx = \frac{-4t}{(t^2-1)^2}dt \end{array}\right] =$$

$$= \int \frac{t-1}{t+1} \cdot \frac{-4t}{(t-1)^2(t+1)^2}dt = \int \frac{-4t}{(t-1)(t+1)^3}dt = (*)$$

$$\frac{-4t}{(t-1)(t+1)^3} = \frac{A}{t-1} + \frac{B}{t+1} + \frac{C}{(t+1)^2} + \frac{D}{(t+1)^3} \quad \Big/ \cdot (t-1)(t+1)^3$$

$$-4t = A(t+1)^3 + B(t-1)(t+1)^2 + C(t-1)(t+1) + D(t-1)$$

$$-4t = At^3 + 3At^2 + 3At + 3A + Bt^3 + Bt^2 - Bt - B + Ct^2 - C + Dt - D$$

$$-4t = t^3(A+B) + t^2(3A+B+C) + t(3A-B+D) + 3A - B - C - D$$

$$\left.\begin{array}{ll} \text{уз } 1: & 0 = 3A - B - C - D \\ \text{уз } t: & -4 = 3A - B + D \\ \text{уз } t^2: & 0 = 3A + B + C \\ \text{уз } t^3: & 0 = A + B \end{array}\right\} \left(\begin{array}{c}\text{рачун} \\ \dots\end{array}\right) \quad \begin{array}{l} A = -\frac{2}{5} \\ B = \frac{2}{5} \\ C = \frac{4}{5} \\ D = -\frac{12}{5} \end{array}$$

$$(\star) = -\frac{2}{5}\int\frac{dt}{t-1} + \frac{2}{5}\int\frac{dt}{t+1} + \frac{4}{5}\int\frac{dt}{(t+1)^2} - \frac{12}{5}\int\frac{dt}{(t+1)^3} =$$

$$= -\frac{2}{5}\ln|t-1| + \frac{2}{5}\ln|t+1| - \frac{4}{5}\cdot\frac{1}{t+1} + \frac{6}{5}\cdot\frac{1}{(t+1)^2} + C =$$

$$= \frac{2}{5}\ln\left|\frac{t+1}{t-1}\right| - \frac{4t-2}{5(t+1)^2} + C =$$

$$= \frac{2}{5}\ln\left|\frac{\sqrt{\frac{x+1}{x-1}}+1}{\sqrt{\frac{x+1}{x-1}}-1}\right| - \frac{4\sqrt{\frac{x+1}{x-1}}-2}{5\left(\sqrt{\frac{x+1}{x-1}}+1\right)^2} + C$$

може да се ради као интеграл рационалне ф-је

(б) $$\int\frac{\sqrt{x-2}+1}{\sqrt{x-2}-1}\,dx = \left[\begin{array}{l} t^2 = x-2 \\ dx = 2t\,dt \end{array}\right] = \int\frac{t+1}{t-1}\cdot 2t\,dt =$$

$$= 2\int\frac{t^2-1+1+t-1+1}{t-1}\,dt = 2\int\frac{(t-1)(t+1)+t-1+2}{t-1}\,dt =$$

$$= 2\int\left(t+2+\frac{2}{t-1}\right)dt = t^2+4t+4\ln|t-1| + C =$$

$$= x-2+4\sqrt{x-2}+4\ln|\sqrt{x-2}-1| + C \quad \blacksquare$$

Интеграли облика $\int R\left(x, \sqrt[n_1]{\frac{ax+b}{cx+d}}, \sqrt[n_2]{\frac{ax+b}{cx+d}}, \dots, \sqrt[n_k]{\frac{ax+b}{cx+d}}\right)dx$

$n = НЗС(n_1, n_2, \dots, n_k)$

смена: $\quad t^n = \frac{ax+b}{cx+d}$

31. (а) $\int \frac{\sqrt{x}}{1+\sqrt[3]{x}}\,dx$; (б) $\int \frac{\sqrt{x}}{1+\sqrt[4]{x}}\,dx$

решење

(а)
$$\int \frac{\sqrt{x}}{1+\sqrt[3]{x}}\,dx = \begin{bmatrix} t^6 = x \\ 6t^5dt = dx \end{bmatrix} = \int \frac{\sqrt{t^6}}{1+\sqrt[3]{t^6}} \cdot 6t^5dt =$$

$$= 6\int \frac{t^8-1+1}{1+t^2}\,dt = 6\int \frac{(t^4-1)(t^4+1)+1}{1+t^2}\,dt =$$

$$= 6\int \frac{(t^2-1)(t^2+1)(t^4+1)+1}{1+t^2}\,dt =$$

$$= 6\int \left((t^2-1)(t^4+1) + \frac{1}{1+t^2}\right)dt =$$

$$= 6\int \left(t^6-t^4+t^2-1+\frac{1}{1+t^2}\right)dt =$$

$$= \frac{6}{7}t^7 - \frac{6}{5}t^5 + 2t^3 - 6t + 6\,\mathrm{arctg}\,t + C =$$

$$= \frac{6}{7}x^{\frac{7}{6}} - \frac{6}{5}x^{\frac{5}{6}} + 2\sqrt{x} - 6\sqrt[6]{x} + 6\,\mathrm{arctg}\,\sqrt[6]{x} + C$$

б) $\int \frac{\sqrt{x}}{1+\sqrt[4]{x}}\,dx = \begin{bmatrix} t^4 = x \\ 4t^3dt = dx \end{bmatrix} = \int \frac{\sqrt{t^4}}{1+\sqrt[4]{t^4}} \cdot 4t^3dt =$

$= 4\int \frac{t^5}{1+t}\,dt = \begin{bmatrix} 1+t = p \\ dt = dp \end{bmatrix} = 4\int \frac{(p-1)^5}{p}\,dp =$

$= 4\int \frac{p^5-5p^4+10p^3-10p^2+5p-1}{p}\,dp =$

$= 4\int \left(p^4-5p^3+10p^2-10p+5-\frac{1}{p}\right)dp =$

$= \frac{4}{5}p^5 - 5p^4 + \frac{40}{3}p^3 - 20p^2 + 20p - 4\ln|p| + C =$

$= \frac{4}{5}\left(\sqrt[4]{x}+1\right)^5 - 5\left(\sqrt[4]{x}+1\right)^4 + \frac{40}{3}\left(\sqrt[4]{x}+1\right)^3 - 20\left(\sqrt[4]{x}+1\right)^2 + 20\left(\sqrt[4]{x}+1\right) -$

$- 4\ln\left|\sqrt[4]{x}+1\right| + C$ ■

Чебишевљеви интеграли $\int (ax^n+b)^p \cdot x^m\,dx$

1. корак: $\boxed{\text{смена } x^n = t}$, $dx = \frac{1}{n} t^{\frac{1}{n}-1} dt$

$$\int (ax^n+b)^p x^m\,dx = \left[\begin{array}{l} t = x^n \\ x = t^{\frac{1}{n}} \\ dx = \frac{1}{n} t^{\frac{1}{n}-1} dt \end{array}\right] =$$

$$= \int (at+b)^p \cdot t^{\frac{m}{n}} \cdot \frac{1}{n} \cdot t^{\frac{1}{n}-1}\,dt =$$

$$= \frac{1}{n} \int (at+b)^p \cdot t^{\frac{m-n+1}{n}}\,dt =$$

означимо $q := \frac{m-n+1}{n}$

$$= \frac{1}{n} \int (at+b)^p \cdot t^q\,dt$$

2. корак: рачунање $\int (at+b)^p \cdot t^q\,dt$

$\boxed{1^\circ}$ $p \in \mathbb{Z}$

$q \in \mathbb{Q}$, $q = \frac{r}{s}$, $\boxed{\text{смена: } t = y^s}$

$\boxed{2^\circ}$ $q \in \mathbb{Z}$

$p \in \mathbb{Q}$, $p = \frac{r}{s}$, $\boxed{\text{смена: } at+b = y^s}$

$\boxed{3^\circ}$ $p+q \in \mathbb{Z}$

$p \in \mathbb{Q}$, $p = \frac{r}{s}$, $\boxed{\text{смена: } \frac{at+b}{t} = y^s}$

$\boxed{32.}$ (a) $\displaystyle\int \frac{\sqrt{x}}{(1+\sqrt[3]{x})^2}\,dx$; (б) $\displaystyle\int \sqrt{x^3+x^4}\,dx$;

(в) $\displaystyle\int \frac{dx}{(x+1)\sqrt{x^2+x}}$

решење

(a) $$\int \frac{\sqrt{x}}{(1+\sqrt[3]{x})^2}\,dx = \int \left(1+x^{\frac{1}{3}}\right)^{-2} \cdot x^{\frac{1}{2}}\,dx =$$

$$= \begin{bmatrix} t = x^{\frac{1}{3}} \\ x = t^3 \\ dx = 3t^2\,dt \end{bmatrix} = 3\int (1+t)^{-2} \cdot t^{\frac{3}{2}} \cdot t^2\,dt =$$

$\boxed{1^\circ\ p \in \mathbb{Z}}$

$$= 3\int (1+t)^{-2} \cdot t^{\frac{7}{2}}\,dt = \begin{bmatrix} t = y^2 \\ dt = 2y\,dy \end{bmatrix} =$$

$\boxed{\text{рационална фја}}$

$$= 3\int \frac{y^7}{(1+y^2)^2} \cdot 2y\,dy = 6\int \frac{y^8}{(1+y^2)^2}\,dy =$$

$$y^8 : (y^4+2y^2+1) = y^4 - 2y^2 + 3$$

$$\begin{array}{l}
\underline{-y^8 \pm 2y^6 \pm y^4} \\
\quad -2y^6 - y^4 \\
\quad \underline{-2y^6 - 4y^4 - 2y^2} \\
\qquad\quad + \quad\ + \quad\ + \\
\qquad 3y^4 + 2y^2 \\
\qquad \underline{3y^4 + 6y^2 + 3} \\
\qquad\ - \quad\ - \quad - \\
\qquad\quad -4y^2 - 3
\end{array}$$

$$\frac{y^8}{(1+y^2)^2} = y^4 - 2y^2 + 3 - \frac{4y^2+3}{(y^2+1)^2}$$

$$\circledast = 6\int\left(y^4 - 2y^2 + 3 - \frac{4y^2+3}{y^2+1}\right) dy =$$

$$= \frac{6}{5}y^5 - 4y^3 + 18y - 6\int \frac{4(y^2+1)}{(y^2+1)^2} dy + 6\int \frac{dy}{(y^2+1)^2} =$$

$$= \frac{6}{5}y^5 - 4y^3 + 18y - 24\operatorname{arctg} y + \frac{3y}{y^2+1} + 3\operatorname{arctg} y + C = \dots$$

(δ) $$\int \sqrt{x^3 + x^4}\, dx = \int \sqrt{x^4(x^{-1}+1)}\, dx =$$

$$= \int (x^{-1}+1)^{\frac{1}{2}} \cdot x^2 dx = \begin{bmatrix} x^{-1} = t \\ x = \frac{1}{t} \\ dx = -\frac{1}{t^2}dt \end{bmatrix} =$$

$$= \int (t+1)^{\frac{1}{2}} \cdot t^{-2} \cdot (-1) \cdot t^{-2} dt =$$

$$= -\int (t+1)^{\frac{1}{2}} \cdot t^{-4}\, dt = \begin{bmatrix} t+1 = y^2 \\ dt = 2y\,dy \end{bmatrix} =$$

$$= -\int y \cdot \frac{1}{(y^2-1)^4} \cdot 2y\,dy = -2\int \frac{y^2}{(y^2-1)^4} dy = \circledast$$

$$\frac{y^2}{(y-1)^4(y+1)^4} = \frac{A}{y+1} + \frac{B}{(y+1)^2} + \frac{C}{(y+1)^3} + \frac{D}{(y+1)^4} + \frac{E}{y-1} + \frac{F}{(y-1)^2} + \frac{G}{(y-1)^3} + \frac{H}{(y-1)^4}$$

... рачун ...

$$A = B = F = -\frac{1}{32}$$

$$E = \frac{1}{32}$$

$$D = H = \frac{1}{16}$$

$$C = G = 0$$

$$\star = \frac{1}{16}\ln|y+1| - \frac{1}{16}\cdot\frac{1}{y+1} - \frac{1}{24}\cdot\frac{1}{(y+1)^3} - \frac{1}{16}\ln|y-1| - \frac{1}{16}\cdot\frac{1}{y-1}$$

$$- \frac{1}{24}\cdot\frac{1}{(y-1)^3} + C =$$

$$= \frac{1}{16}\ln\sqrt{\frac{1}{x}+1} - \frac{1}{16}\cdot\frac{1}{\sqrt{\frac{1}{x}+1}+1} - \frac{1}{24}\cdot\frac{1}{\left(\sqrt{\frac{1}{x}+1}+1\right)^3} - \frac{1}{16}\ln\left|\sqrt{\frac{1}{x}+1}-1\right| -$$

$$- \frac{1}{16}\cdot\frac{1}{\sqrt{\frac{1}{x}+1}-1} - \frac{1}{24}\cdot\frac{1}{\left(\sqrt{\frac{1}{x}+1}-1\right)^3} + C$$

$\boxed{p+q \in \mathbb{Z}}$

$$(\text{б})\ \int\frac{dx}{(x+1)\sqrt{x^2+x}} = \int\frac{dx}{(x+1)\cdot\sqrt{x}\cdot\sqrt{x+1}} = \int (x+1)^{-\frac{3}{2}}\cdot x^{-\frac{1}{2}}\,dx =$$

$$= \left[\begin{array}{l} \frac{x+1}{x} = y^2 \\ x+1 = xy^2 \\ x = \frac{1}{y^2-1} \\ dx = \frac{-2y\,dy}{(y^2-1)^2} \end{array}\right] = \int\left(\frac{1}{y^2-1}+1\right)^{-\frac{3}{2}}\cdot\left(\frac{1}{y^2-1}\right)^{-\frac{1}{2}}\cdot\frac{-2y}{(y^2-1)^2}\,dy =$$

$$= \int \frac{(y^2-1)^{\frac{3}{2}}}{y^3} \cdot (y^2-1)^{\frac{1}{2}} \cdot \frac{-2y}{(y^2-1)^2}\, dy =$$

$$= -2 \int \frac{dy}{y^2} = \frac{2}{y} + C = 2\sqrt{\frac{x}{x+1}} + C$$ ■

33. (a) $\int \frac{dx}{x\sqrt[3]{1+x^5}}$; (б) $\int \frac{\sqrt[3]{1+\sqrt[4]{x}}}{\sqrt{x}}\, dx$

решење

(a) $$\int \frac{dx}{x\sqrt[3]{1+x^5}} = \int (x^5+1)^{-\frac{1}{3}} \cdot x^{-1}\, dx = \left[\begin{array}{l} t = x^5 \\ dx = \frac{1}{5} t^{-\frac{4}{5}} dt \end{array}\right] =$$

$$= \int (t+1)^{-\frac{1}{3}} \cdot t^{-\frac{1}{5}} \cdot \frac{1}{5} t^{-\frac{4}{5}}\, dt = \int (t+1)^{-\frac{1}{3}} \cdot t^{-1}\, dt =$$

$$= \left[\begin{array}{l} t+1 = y^3 \\ dt = 3y^2 dy \end{array}\right] = \int y^{-1} \cdot (y^3-1)^{-1} \cdot 3y^2\, dy =$$

интеграција рац. ф-је

$$= 3\int \frac{y}{y^3-1}\, dy = \dots = -\frac{1}{2}\ln(y^2+y+1) + \ln|1-y| +$$

$$+ \sqrt{3}\, \mathrm{arctg}\left(\frac{2y+1}{\sqrt{3}}\right) + C =$$

$$= -\frac{1}{2}\ln\left((x^5+1)^{\frac{2}{3}} + (x^5+1)^{\frac{1}{3}} + 1\right) + \ln|1-(x^5+1)^{\frac{1}{3}}| +$$

$$+ \sqrt{3}\, \mathrm{arctg}\left(\frac{2(x^5+1)^{\frac{1}{3}}+1}{\sqrt{3}}\right) + C$$

(δ) $\displaystyle\int \frac{\sqrt[3]{1+\sqrt[4]{x}}}{\sqrt{x}}\,dx = \int \left(x^{\frac{1}{4}}+1\right)^{\frac{1}{3}} \cdot x^{-\frac{1}{2}}\,dx =$

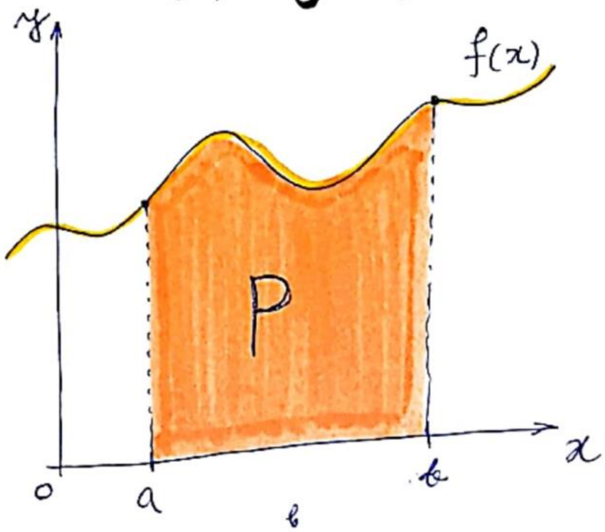
$= \begin{bmatrix} x^{\frac{1}{4}} = t \\ dx = 4t^3 dt \end{bmatrix} = \displaystyle\int (t+1)^{\frac{1}{3}} \cdot t^{-2} \cdot 4t^3\,dt =$

$= 4 \displaystyle\int (t+1)^{\frac{1}{3}} \cdot t\,dt = \begin{bmatrix} t+1 = y^3 \\ dt = 3y^2 dy \end{bmatrix} =$

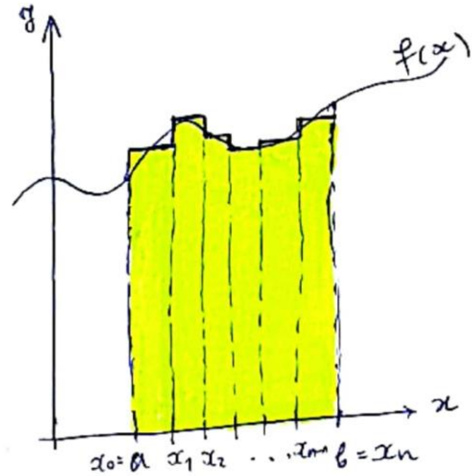
$= 4 \displaystyle\int y \cdot (y^3-1) \cdot 3y^2\,dy = 12 \int (y^6 - y^3)\,dy =$

$= \frac{12}{7} y^7 - 3y^4 + C = \frac{12}{7}\left(x^{\frac{1}{4}}+1\right)^{\frac{7}{3}} - 3\left(x^{\frac{1}{4}}+1\right)^{\frac{4}{3}} + C$ ∎

Одређени интеграли



$$P = \int_a^b f(x)\,dx$$



$$\sigma_n = \sum_{i=1}^{n} (x_i - x_{i-1}) \cdot f(\xi_i)$$

$$\xi_i \in (x_{i-1}, x_i)$$

$$\int_a^b f(x)\,dx = \lim \sigma_n$$

лимес по све мањим поделама интервала $[a, b]$

Став

(1) $\displaystyle\int_a^b (\alpha f(x) + \beta g(x))\,dx = \alpha \int_a^b f(x)\,dx + \beta \int_a^b g(x)\,dx$;

(2) $\displaystyle\int_a^a f(x)\,dx = 0$

(3) $\displaystyle\int_a^b f(x)\,dx = - \int_b^a f(x)\,dx$;

(4) $\displaystyle\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx, \quad c \in (a, b)$;

(5) $\displaystyle\left| \int_a^b f(x)\,dx \right| \leq \int_a^b |f(x)|\,dx$.

Теорема (Њутн - Лајбниц)

Нека је $f:[a,b]\to\mathbb{R}$ интеграбилна и $F:[a,b]\to\mathbb{R}$ њена примитивна функција. Тада је

$$\int_a^b f(x)\,dx = F(b)-F(a).$$

Теорема (Смена променљиве)

$f:[a,b]\to\mathbb{R}$ интеграбилна, $\varphi:[\alpha,\beta]\to[a,b]$ бијекција.

Тада $$\int_a^b f(x)\,dx = \int_\alpha^\beta f(\varphi(t))\cdot\varphi'(t)\,dt.$$

Теорема (Парцијална интеграција)

$u,v:[a,b]\to\mathbb{R}$ непрекидно - диференцијабилне

$$\int_a^b u\,dv = u\cdot v\Big|_a^b - \int_a^b v\,du$$

$$\boxed{u\cdot v\big|_a^b = u(b)v(b)-u(a)v(a)}$$

Теорема

Ако је f непрекидна, онда има примитивну функцију дату са $F(x)=\int_a^x f(t)\,dt$.

$\boxed{1.}$ (а) $\int_0^1 e^{2x}\,dx$; (б) $\int_0^e \frac{e^x+1}{e^x+x}\,dx$; (в) $\int_1^2 \frac{dx}{2x(x^2+1)}$

решење

(а) $$\int_0^1 e^{2x}dx = \left[\begin{array}{l} t=2x \\ dx=\frac{1}{2}\,dt \\ x=0 \rightsquigarrow t=0 \\ x=1 \rightsquigarrow t=2 \end{array}\right] = \frac{1}{2}\int_0^2 e^t dt = \frac{1}{2}e^t\Big|_0^2 = \frac{1}{2}(e^2-e^0) = \frac{1}{2}(e^2-1)$$

(б) $$\int_0^e \frac{e^x+1}{e^x+x}\,dx = \left[\begin{array}{l} t=e^x+x \\ dt=(e^x+1)dx \\ x=0 \rightarrow t=1 \\ x=e \rightarrow t=e^e+e \end{array}\right] = \int_1^{e^e+e}\frac{dt}{t} = \ln|t|\,\Big|_1^{e^e+e} =$$

$$= \ln(e^e+e)+\ln 1 = \ln(e^e+e)$$

(в) $$\int_1^2 \frac{dx}{2x(x^2+1)} = \int_1^2 \frac{x\,dx}{2x^2(x^2+1)} = \left[\begin{array}{l} t=x^2 \\ dt=2x\,dx \\ x=1 \rightarrow t=1 \\ x=2 \rightarrow t=4 \end{array}\right] = \frac{1}{4}\int_1^4 \frac{dt}{t(t+1)} =$$

$$= \frac{1}{4}\int_1^4 \frac{1+t-t}{t(t+1)}\,dt = \frac{1}{4}\int_1^4\left(\frac{1}{t}-\frac{1}{t+1}\right)dt = \left(\frac{1}{4}\ln|t|-\frac{1}{4}\ln|t+1|\right)\Big|_1^4 =$$

$$= \frac{1}{4}\ln 4-\frac{1}{4}\ln 5-\frac{1}{4}\ln 1+\frac{1}{4}\ln 2 = \frac{3}{4}\ln 2-\frac{1}{4}\ln 5 \quad \blacksquare$$