$\boxed{9.}$ (a) $\int x^2 \arcsin x \, dx$ (ЈАН1 2021.) ; (б) $\int \frac{\arcsin x}{x^2} dx$

решење

(a) $\int x^2 \arcsin x \, dx = \begin{bmatrix} u = \arcsin x & dv = x^2 dx \\ du = \frac{1}{\sqrt{1-x^2}} dx & v = \frac{1}{3} x^3 \end{bmatrix} =$

$$= \frac{1}{3} x^3 \arcsin x - \frac{1}{3} \int \frac{x^3}{\sqrt{1-x^2}} dx =$$

$$= \frac{1}{3} x^3 \arcsin x - \frac{1}{3} \int \frac{x^2 \cdot x}{\sqrt{1-x^2}} dx = \begin{bmatrix} 1 - x^2 = t \\ x^2 = 1 - t \\ 2x dx = -dt \\ x dx = -\frac{1}{2} dt \end{bmatrix} =$$

$$= \frac{1}{3} x^3 \arcsin x - \frac{1}{3} \int \frac{(1-t) \cdot (-\frac{1}{2})}{\sqrt{t}} dt =$$

$$= \frac{1}{3} x^3 \arcsin x + \frac{1}{6} \int \left(\frac{1}{\sqrt{t}} - \sqrt{t} \right) dt =$$

$$= \frac{1}{3} x^3 \arcsin x + \frac{1}{3} \sqrt{1-x^2} - \frac{1}{9} (1-x^2)^{\frac{3}{2}} + c$$

(б) $\int \frac{\arcsin x}{x^2}\,dx = \left[\begin{array}{ll} u=\arcsin x & dv=\frac{1}{x^2}\,dx \\ du=\frac{1}{\sqrt{1-x^2}}\,dx & v=-\frac{1}{x} \end{array}\right] =$

$$= -\frac{\arcsin x}{x} + \int \frac{1}{x\sqrt{1-x^2}}\,dx =$$

$$= -\frac{\arcsin x}{x} + \int \frac{x}{x^2\sqrt{1-x^2}}\,dx = \left[\begin{array}{l} t^2=1-x^2 \\ x^2=1-t^2 \\ x\,dx=-t\,dt \end{array}\right] =$$

$$= -\frac{\arcsin x}{x} - \int \frac{t\,dt}{(1-t^2)\,t} =$$

$$= -\frac{\arcsin x}{x} - \int \frac{dt}{1-t^2} =$$

$$\boxed{\int \frac{dt}{1-t^2} = \frac{1}{2}\ln\left|\frac{1+t}{1-t}\right| + c}$$

$$= -\frac{\arcsin x}{x} - \frac{1}{2}\ln\left|\frac{1+\sqrt{1-x^2}}{1-\sqrt{1-x^2}}\right| + c$$

$\boxed{10.}$ (а) $\int x^{\alpha}\ln x\,dx$; (б) $\int \ln(2x^2+1)\,dx$.

$\alpha \neq -1$

решење

(а) $\int x^{\alpha}\ln x\,dx = \left[\begin{array}{ll} u=\ln x & dv=x^{\alpha}\,dx \\ du=\frac{1}{x}\,dx & v=\frac{1}{\alpha+1}x^{\alpha+1} \end{array}\right] =$

$$= \frac{x^{\alpha+1}}{\alpha+1}\ln x - \frac{1}{\alpha+1}\int \frac{x^{\alpha+1}}{x}\,dx =$$

$$= \frac{x^{\alpha+1}}{\alpha+1}\ln x - \frac{1}{\alpha+1}\int x^{\alpha}\,dx =$$

$$= \frac{x^{\alpha+1}}{\alpha+1}\ln x - \frac{x^{\alpha+1}}{(\alpha+1)^2} + c$$

(б) $\int \ln(2x^2+1)\,dx = \begin{bmatrix} u=\ln(2x^2+1) & dv=dx \\ du=\frac{4x}{2x^2+1}\,dx & v=x \end{bmatrix} =$

$$= x\ln(2x^2+1) - 2\int \frac{2x^2+1-1}{2x^2+1}\,dx =$$

$$= x\ln(2x^2+1) - 2\int \left(1-\frac{1}{2x^2+1}\right) dx =$$

$$= x\ln(2x^2+1) - 2x + 2\underbrace{\int \frac{1}{2x^2+1}\,dx}_{\boxed{\text{задатак 3.(а) на стр. -4-}}} =$$

$$= x\ln(2x^2+1) - 2x + \sqrt{2}\,\mathrm{arctg}\,(x\sqrt{2}) + c \quad \blacksquare$$

11. (а) $\int e^{\alpha x}\sin\beta x\,dx$; (б) $\int e^{\alpha x}\cos\beta x\,dx$; (в) $\int \cos(\ln x)\,dx$

решење

(а) $\underbrace{\int e^{\alpha x}\sin\beta x\,dx}_{I} = \begin{bmatrix} u=e^{\alpha x} & dv=\sin\beta x\,dx \\ du=\alpha e^{\alpha x}dx & v=-\frac{1}{\beta}\cos\beta x \end{bmatrix} =$

$$= -\frac{1}{\beta}e^{\alpha x}\cos\beta x + \frac{\alpha}{\beta}\int e^{\alpha x}\cos\beta x\,dx = \begin{bmatrix} u=e^{\alpha x} & dv=\cos\beta x\,dx \\ du=\alpha e^{\alpha x}dx & v=\frac{1}{\beta}\sin\beta x \end{bmatrix} =$$

$$= -\frac{1}{\beta}e^{\alpha x}\cos\beta x + \frac{\alpha}{\beta}\left(\frac{1}{\beta}e^{\alpha x}\sin\beta x - \frac{\alpha}{\beta}\underbrace{\int e^{\alpha x}\sin\beta x\,dx}_{I}\right)$$

Спојимо почетак и крај:

$$I = -\frac{1}{\beta}e^{\alpha x}\cos\beta x + \frac{\alpha}{\beta^2}e^{\alpha x}\sin\beta x - \frac{\alpha^2}{\beta^2}I$$

$$I + \frac{\alpha^2}{\beta^2} I = -\frac{1}{\beta} e^{\alpha x}\cos\beta x + \frac{\alpha}{\beta^2} e^{\alpha x}\sin\beta x \Big/ : \left(1+\frac{\alpha^2}{\beta^2}\right)$$

$$I = -\frac{\beta}{\alpha^2+\beta^2} e^{\alpha x}\cos\beta x + \frac{\alpha}{\alpha^2+\beta^2} e^{\alpha x}\sin\beta x + c$$

(в) $$\underbrace{\int e^{\alpha x}\cos\beta x\,dx}_{I} = \left[\begin{array}{ll} u = \cos\beta x & dv = e^{\alpha x}dx \\ du = -\beta\sin\beta x\,dx & v = \frac{1}{\alpha}e^{\alpha x} \end{array}\right] =$$

$$= \frac{1}{\alpha}e^{\alpha x}\cos\beta x + \frac{\beta}{\alpha}\int e^{\alpha x}\sin\beta x\,dx = \left[\begin{array}{ll} u = \sin\beta x & dv = e^{\alpha x}dx \\ du = \beta\cos\beta x\,dx & v = \frac{1}{\alpha}e^{\alpha x} \end{array}\right] =$$

$$= \frac{1}{\alpha}e^{\alpha x}\cos\beta x + \frac{\beta}{\alpha^2}e^{\alpha x}\sin\beta x - \frac{\beta^2}{\alpha^2}\underbrace{\int e^{\alpha x}\cos\beta x\,dx}_{I}$$

$$\Rightarrow I = \frac{1}{\alpha}e^{\alpha x}\cos\beta x + \frac{\beta}{\alpha^2}e^{\alpha x}\sin\beta x - \frac{\beta^2}{\alpha^2} I$$

$$I = \frac{\alpha}{\alpha^2+\beta^2}e^{\alpha x}\cos\beta x + \frac{\beta}{\alpha^2+\beta^2}e^{\alpha x}\sin\beta x + c$$

(г) 1. начин (користејќи (б))

$$\int \cos(\ln x)\,dx = \left[\begin{array}{l} t = \ln x \\ x = e^t \\ dx = e^t dt \end{array}\right] = \int \cos t \cdot e^t dt =$$

$$= \frac{1}{2}e^t\cos t + \frac{1}{2}e^t\sin t + c =$$

$\boxed{\text{део (б) за } \alpha = \beta = 1}$

$$= \frac{1}{2}x\cos(\ln x) + \frac{1}{2}x\sin(\ln x) + c$$

2. начин (без дела (б))

$$\underbrace{\int \cos(\ln x)\,dx}_{I} = \begin{bmatrix} u = \cos(\ln x) & dv = dx \\ du = \frac{-\sin(\ln x)}{x}\,dx & v = x \end{bmatrix} =$$

$$= x\cos(\ln x) + \int \sin(\ln x)\,dx = \begin{bmatrix} u = \sin(\ln x) & dv = dx \\ du = \frac{\cos(\ln x)}{x}\,dx & v = x \end{bmatrix} =$$

$$= x\cos(\ln x) + x\sin(\ln x) - \underbrace{\int \cos(\ln x)\,dx}_{I}$$

Дакле, $I = x\cos(\ln x) + x\sin(\ln x) - I$

$$\Rightarrow I = \frac{1}{2}x\cos(\ln x) + \frac{1}{2}x\sin(\ln x) + c. \quad \blacksquare$$

$\boxed{*}$ $\int \frac{dx}{(x^2+1)^2} = I$

решење 1. начин

$$\int \frac{dx}{x^2+1} = \begin{bmatrix} u = \frac{1}{x^2+1} & dv = dx \\ du = \frac{-2x}{(x^2+1)^2}\,dx & v = x \end{bmatrix} = \frac{x}{x^2+1} + 2\int \frac{x^2}{(x^2+1)^2}\,dx =$$

интеграл из таблице

$$= \frac{x}{x^2+1} + 2\int \frac{x^2+1-1}{(x^2+1)^2}\,dx = \frac{x}{x^2+1} + 2\int \frac{1}{x^2+1}\,dx - 2\underbrace{\int \frac{dx}{(x^2+1)^2}}_{I}$$

Дакле,

$$\operatorname{arctg} x + c_1 = \frac{x}{x^2+1} + 2\operatorname{arctg} x + c_2 - 2I$$

$$\Rightarrow I = \frac{x}{2(x^2+1)} + \frac{1}{2}\operatorname{arctg} x + \underbrace{\frac{c_2 - c_1}{2}}_{\overset{\shortparallel}{c}}$$

$$I = \frac{x}{2(x^2+1)} + \frac{1}{2}\operatorname{arctg} x + c$$

2. начин

$$\int \frac{1}{(x^2+1)^2}\,dx = \int \frac{1+x^2-x^2}{(x^2+1)^2}\,dx = \int \frac{dx}{1+x^2} - \int \frac{x^2}{(x^2+1)^2}\,dx =$$

$$= \operatorname{arctg} x - \int \frac{x \cdot x}{(x^2+1)^2}\,dx = \begin{bmatrix} u = x & dv = \frac{x}{(x^2+1)^2}\,dx \\ du = dx & v = -\frac{1}{2}\cdot\frac{1}{1+x^2} \end{bmatrix}$$

$$= \operatorname{arctg} x - \left(-\frac{1}{2}\cdot\frac{x}{1+x^2} + \frac{1}{2}\int \frac{dx}{1+x^2} \right) =$$

$$= \operatorname{arctg} x + \frac{x}{2(1+x^2)} - \frac{1}{2}\operatorname{arctg} x + c =$$

$$= \frac{x}{2(x^2+1)} + \frac{1}{2}\operatorname{arctg} x + c$$

Коришћено:

$$\int \frac{x}{(x^2+1)^2}\,dx = \begin{bmatrix} t = x^2+1 \\ dt = 2x\,dx \\ x\,dx = \frac{1}{2}\,dt \end{bmatrix} = \frac{1}{2}\int \frac{dt}{t^2} = -\frac{1}{2}\cdot\frac{1}{t} + c =$$

$$= -\frac{1}{2}\cdot\frac{1}{1+x^2} + c$$

$\boxed{12.}$ (a) $\int \sin(\sqrt{1+x})\,dx$; (б) $\int \dfrac{x\ln(x+\sqrt{x^2+1})}{\sqrt{x^2+1}}\,dx$

решење

(a) $\int \sin\sqrt{1+x}\,dx = \begin{bmatrix} 1+x=t^2 \\ dx=2t\,dt \end{bmatrix} = 2\int t\sin t\,dt = \begin{bmatrix} u=t & dv=\sin t\,dt \\ du=dt & v=-\cos t \end{bmatrix}$

$= 2\left(-t\cos t + \int \cos t\,dt\right) = -2t\cos t + 2\sin t + c =$

$= -2\sqrt{1+x}\,\cos\sqrt{1+x} + 2\sin\sqrt{1+x} + c$

(б) $\int \dfrac{x\ln(x+\sqrt{x^2+1})}{\sqrt{x^2+1}}\,dx = \begin{bmatrix} u=\ln(x+\sqrt{x^2+1}) & dv=\dfrac{x}{\sqrt{x^2+1}}\,dx \\ du=\dfrac{1+\frac{2x}{2\sqrt{x^2+1}}}{x+\sqrt{x^2+1}}\,dx & v=\sqrt{x^2+1} \end{bmatrix} =$

$= \sqrt{x^2+1}\cdot\ln(x+\sqrt{x^2+1}) - \int \dfrac{\sqrt{x^2+1}+x}{x+\sqrt{x^2+1}}\,dx =$

$= \sqrt{x^2+1}\cdot\ln(x+\sqrt{x^2+1}) - x + c$ ■

$\boxed{13.}$ (a) $I_n = \int \sin^n x\,dx$; (б) $I_n = \int \cos^n x\,dx$.

решење

(a) $I_n = \int \sin^n x\,dx = \int \sin^{n-1} x\cdot\sin x\,dx =$

$= \begin{bmatrix} u=\sin^{n-1}x & dv=\sin x\,dx \\ du=(n-1)\sin^{n-2}x\cdot\cos x\,dx & v=-\cos x \end{bmatrix} =$

$$= -\cos x \cdot \sin^{n-1} x + (n-1) \int \sin^{n-2} x \cdot \cos^2 x \, dx =$$

$$= -\cos x \cdot \sin^{n-1} x + (n-1) \int \sin^{n-2} x \cdot (1 - \sin^2 x) \, dx =$$

$$= -\cos x \cdot \sin^{n-1} x + (n-1) \underbrace{\int \sin^{n-2} x \, dx}_{I_{n-2}} - (n-1) \underbrace{\int \sin^n x \, dx}_{I_n}$$

Спојимо почетак и крај:

$$I_n = -\cos x \cdot \sin^{n-1} x + (n-1) I_{n-2} - (n-1) I_n$$

$$\boxed{I_n = -\frac{1}{n} \cos x \cdot \sin^{n-1} x + \frac{n-1}{n} I_{n-2}}$$

Одавде можемо одредити I_n за свако $n \in \mathbb{N}$.

Нпр. $n = 5$.

$$I_5 = -\frac{1}{5} \cos x \cdot \sin^4 x + \frac{4}{5} I_3 =$$

$$= -\frac{1}{5} \cos x \cdot \sin^4 x + \frac{4}{5} \Big(-\frac{1}{3} \cos x \cdot \sin^2 x + \frac{2}{3} \underbrace{I_1}_{\int \sin x dx} \Big) =$$

$$= -\frac{1}{5} \cos x \cdot \sin^4 x - \frac{4}{15} \cos x \cdot \sin^2 x - \frac{8}{15} \cos x + c$$

Нпр. $n = 4$

$$I_4 = -\frac{1}{4} \cos x \cdot \sin^3 x + \frac{3}{4} \underbrace{I_2}_{\int \sin^2 x dx} = -\frac{1}{4} \cos x \cdot \sin^3 x + \frac{3}{4} \int \frac{1 - \cos 2x}{2} dx =$$

$$= -\frac{1}{4} \cos x \cdot \sin^3 x + \frac{3}{8} x - \frac{3}{16} \sin 2x + c$$

(б) $I_n = \int \cos^{n-1} x \cdot \cos x \, dx = \left[\begin{array}{ll} u = \cos^{n-1} x & dv = \cos x \, dx \\ du = -(n-1)\cos^{n-2} x \cdot \sin x \, dx & v = \sin x \end{array} \right]$

$$= \cos^{n-1} x \cdot \sin x + (n-1) \int \cos^{n-2} x \cdot (1-\cos^2 x) \, dx =$$

$$= \cos^{n-1} x \cdot \sin x + (n-1) I_{n-2} - (n-1) I_n$$

Даке, $\boxed{I_n = \frac{1}{n} \cos^{n-1} x \cdot \sin x + \frac{n-1}{n} I_{n-2}}$ ■

$\boxed{14.}$ $\int \frac{x e^{\operatorname{arctg} x}}{(1+x^2)^{\frac{3}{2}}} \, dx$

решење

$$\underbrace{\int \frac{x e^{\operatorname{arctg} x}}{(1+x^2)^{\frac{3}{2}}} \, dx}_{I} = \left[\begin{array}{ll} u = \frac{x}{\sqrt{x^2+1}} & dv = \frac{e^{\operatorname{arctg} x}}{1+x^2} dx \\ du = \frac{\sqrt{x^2+1} - \frac{2x \cdot x}{2\sqrt{x^2+1}}}{x^2+1} dx & v = e^{\operatorname{arctg} x} \end{array} \right]$$

$$= \frac{x e^{\operatorname{arctg} x}}{\sqrt{x^2+1}} - \int e^{\operatorname{arctg} x} \cdot \frac{x^2+1-x^2}{(x^2+1)^{\frac{3}{2}}} \, dx =$$

$$= \left[\begin{array}{ll} u = \frac{1}{\sqrt{1+x^2}} & dv = \frac{e^{\operatorname{arctg} x}}{1+x^2} dx \\ du = \frac{-x}{(1+x^2)^{\frac{3}{2}}} dx & v = e^{\operatorname{arctg} x} \end{array} \right] =$$

$$= \frac{x e^{\operatorname{arctg} x}}{\sqrt{x^2+1}} - \frac{e^{\operatorname{arctg} x}}{\sqrt{x^2+1}} - \underbrace{\int \frac{x e^{\operatorname{arctg} x}}{(x^2+1)^{\frac{3}{2}}} \, dx}_{I}$$

Спојимо почетак и крај:

$$I = \frac{(x-1)\,e^{\operatorname{arctg} x}}{\sqrt{x^2+1}} - I$$

$$\Rightarrow \; I = \frac{(x-1)\,e^{\operatorname{arctg} x}}{2\sqrt{x^2+1}} + c \quad \blacksquare$$

$\boxed{15.}$ (а) $\int \sin 3x \cdot \cos 5x \, dx$; (б) $\int \cos 17x \cdot \cos 12x \, dx$

(в) $\int \sin 4x \cdot \sin 7x \, dx$

решење

(а) $$\int \sin 3x \cdot \cos 5x \, dx = \frac{1}{2}\int \left(\sin 8x + \sin(-2x)\right) dx =$$

$$= -\frac{1}{2}\cdot\frac{1}{8}\cos 8x + \frac{1}{2}\cdot\frac{1}{2}\cos 2x + c = -\frac{1}{16}\cos 8x + \frac{1}{4}\cos 2x + c$$

(б) $$\int \cos 17x \cos 12x \, dx = \frac{1}{2}\int (\cos 29x + \cos 5x)\, dx =$$

$$= \frac{1}{58}\sin 29x + \frac{1}{10}\sin 5x + c$$

(в) $$\int \sin 4x \sin 7x \, dx = \frac{1}{2}\int (\cos(-3x) - \cos 11x)\, dx =$$

$$= \frac{1}{6}\sin 3x - \frac{1}{22}\sin 11x + c \quad \blacksquare$$

Интеграција рационалних функција

Теорема Свака рационална функција се може представити као збир коначно много сабирака облика

① полином

② $\frac{a}{(x-\alpha)^k}$

③ $\frac{ax+b}{(px^2+qx+r)^m}$, где полином px^2+qx+r нема реалних решења (тј. $D=q^2-4pr<0$)

Како решавамо $\int \frac{P_m(x)}{Q_n(x)}dx$?

1. корак: ако је $m \geq n$ поделимо $P_m(x):Q_n(x)$ и запишемо у облику $\frac{P_m(x)}{Q_n(x)} = \text{КОЛИЧНИК} + \frac{\text{ОСТАТАК}}{Q_n(x)}$

2. корак: раставимо $Q_n(x)$ на чиниоце

3. корак: запишемо $\frac{\text{ОСТАТАК}}{Q_n(x)}$ (односно $\frac{P_m(x)}{Q_n(x)}$ ако је $m<n$) као збир сабирака облика ② и ③ и интегралимо сваки посебно.

Пример - за 3. корак

(1) $$\frac{3x^2-5x+2}{(x+1)(x+2)^3(x^2+1)(x^2+x+1)^2} = \frac{A}{x+1} + \frac{B}{x+2} + \frac{C}{(x+2)^2} + \frac{D}{(x+2)^3} +$$

бројилац не утиче на десну страну

$$+ \frac{Ex+F}{x^2+1} + \frac{Gx+H}{x^2+x+1} + \frac{Ix+J}{(x^2+x+1)^2}$$

сада би помножили израз са $(x+1)(x+2)^3(x^2+1)(x^2+x+1)^2$ и нашли константе $A, B, C, \dots, J$.

(2) $$\frac{1}{(x-1)(x+1)} = \frac{A}{x-1} + \frac{B}{x+1} \quad / \cdot (x-1)(x+1)$$

$$1 = A(x+1) + B(x-1)$$

$$1 = x(A+B) + A - B$$

изједначимо шта је уз 1 и уз x

$$\left.\begin{aligned} \text{уз } 1: \quad 1 &= A - B \\ \text{уз } x: \quad 0 &= A + B \end{aligned}\right\} \text{ решимо систем}$$

$$A = \frac{1}{2}, \quad B = -\frac{1}{2}$$

$$\Rightarrow \frac{1}{(x-1)(x+1)} = \frac{\frac{1}{2}}{x-1} + \frac{-\frac{1}{2}}{x+1}$$

16. (а) $\int \frac{5x-3}{x^2-2x-3}\,dx$; (б) $\int \frac{x}{(x+1)^2}\,dx$

решење

Раставимо x^2-2x-3 на чиниоце: $x_{1/2} = \frac{2 \pm \sqrt{4+12}}{2}$

$\Rightarrow x_1 = 3, \; x_2 = -1$

$$\Rightarrow \quad x^2-2x-3=(x-3)(x+1).$$

$$\frac{5x-3}{(x-3)(x+1)}=\frac{A}{x-3}+\frac{B}{x+1} \quad \Big/ \cdot (x-3)(x+1)$$

$$5x-3=A(x+1)+B(x-3)$$

$$5x-3=x(A+B)+A-3B$$

уз x: $\quad 5=A+B \Rightarrow B=5-A$

уз 1: $\quad -3=A-3B$ ← убацимо

$$-3=A-3(5-A) \Rightarrow -3=4A-15 \Rightarrow \boxed{A=3} \Rightarrow \boxed{B=2}$$

Сада се враћамо на интеграл:

$$\int\frac{5x-3}{(x-3)(x+1)}dx=\int\frac{3}{x-3}dx+\int\frac{2}{x+1}=3\ln|x-3|+2\ln|x+1|+c$$

(б) $$\frac{x}{(x+1)^2}=\frac{A}{x+1}+\frac{B}{(x+1)^2} \quad \Big/ \cdot (x+1)^2$$

$$x=Ax+A+B$$

$$\left.\begin{aligned} &\text{уз } x: \quad 1=A \\ &\text{уз } 1: \quad 0=A+B \end{aligned}\right\} \Rightarrow \boxed{\begin{array}{c} A=1 \\ \hline B=-1 \end{array}}$$

решава се сменом $t=x+1$

$$\int\frac{x}{(x+1)^2}dx=\int\frac{1}{x+1}dx-\int\frac{1}{(x+1)^2}dx=$$

$$=\ln|x+1|+\frac{1}{x+1}+c. \quad \blacksquare$$