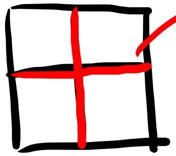
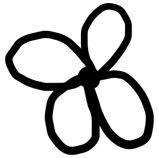
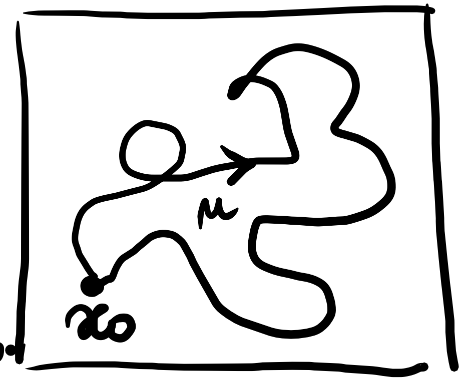
пр. X  $\simeq X/A$ 

$A = + \sim *$, A затв.

Фундаментална група



X – т.п. $x_0 \in X$ базна тачка

(X, x_0) – простор са базном тачком

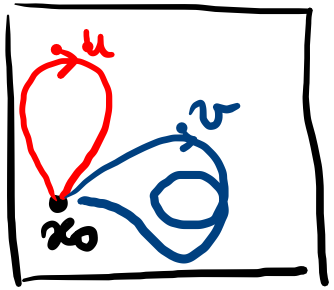
$u: I \to X$ је ПЕТЉА у т. x_0 ако је то пут који спаја x_0 са собом

тј. $u(0) = u(1) = x_0$

$$\boxed{\mathcal{P}(X, x_0)} := \{u: I \to X \mid u \text{ је непрекидно},\ u(0) = u(1) = x_0\}$$

НАДОВЕЗИВАЊЕ ПЕТЉИ $\quad (u, v) \longmapsto u \cdot v$

$u, v \in \mathcal{P}(X, x_0)$



$u \cdot v : I \to X$

$$u \cdot v(t) = \begin{cases} u(2t), & t \in [0, \frac{1}{2}] \\ v(2t-1), & t \in [\frac{1}{2}, 1] \end{cases}$$

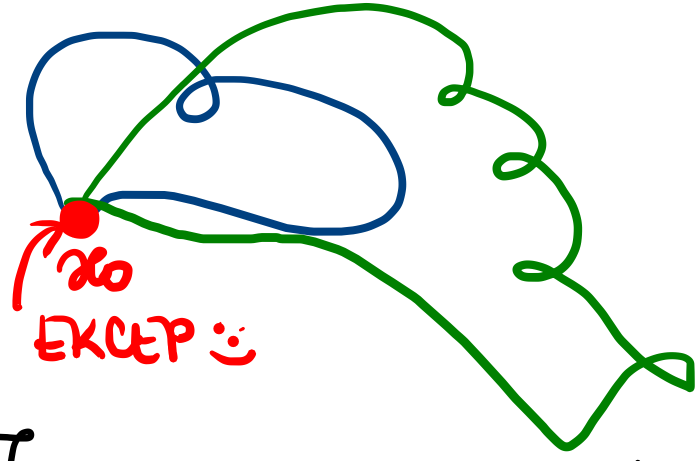
Хомотопне петље, rel$\{0,1\}$

Став $u, u', v, v' \in P(X, x_0)$ $\quad$ $(u, u', v, v' : I \to X)$

$u \simeq u' \ (\text{rel}\{0,1\})$

$v \simeq v' \ (\text{rel}\{0,1\})$

$\Rightarrow \ u \cdot v \simeq u' \cdot v' \ (\text{rel}\{0,1\})$



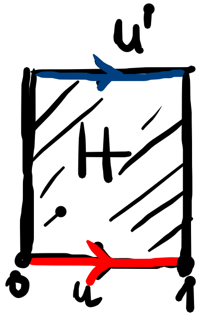
Доказ: $H : I \times I \to X$

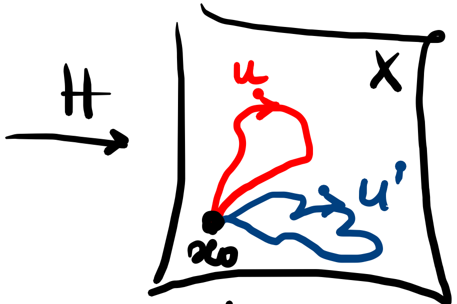
$H : u \simeq u' (\text{rel}\{0,1\})$

$\forall s \in I \quad H(s,0) = u(s)$

$\qquad\qquad H(s,1) = u'(s)$

$\forall t \in I \quad H(0,t) = H(1,t) = x_0$





$u \cdot v(s) = \begin{cases} u(2s), & s \in [0, \frac{1}{2}] \\ v(2s-1), & s \in [\frac{1}{2}, 1] \end{cases}$

Аналогно:

$G : v \simeq v' (\text{rel}\{0,1\})$

$G(s,0) = v(s)$

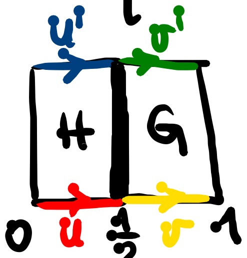
$G(s,1) = v'(s)$

$G(0,t) = G(1,t) = x_0$

$\forall t \in I$

Конструишемо:

$F : I \times I \to X$



$F(s,t) = \begin{cases} H(2s, t), & s \in [0, 1/2] \\ G(2s-1, t), & s \in [\frac{1}{2}, 1] \end{cases}$

* добро деф. $s = 1/2$, $t \in I$

$H(1,t) = x_0 = G(0,t)$ ✓

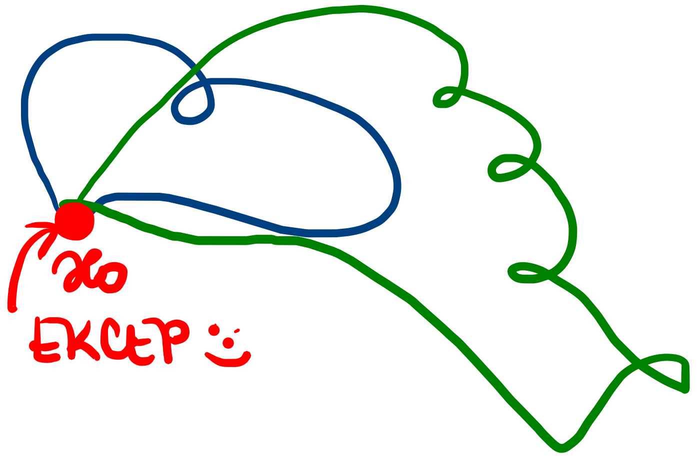
хомотопне петље, rel{0,1}

Став $u, u', v, v' \in P(X, x_0)$ $\quad (u, u', v, v' : I \to X)$

$u \simeq u' \ (rel\{0,1\})$

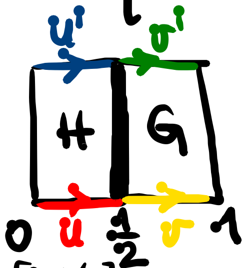
$v \simeq v' \ (rel\{0,1\})$

$\Rightarrow \ u \cdot v \simeq u' \cdot v' \ (rel\{0,1\})$



$u \cdot v(s) = \begin{cases} u(2s), & s \in [0, \frac{1}{2}] \\ v(2s-1), & s \in [\frac{1}{2}, 1] \end{cases}$

Конструишемо:
$F : I \times I \to X$



$F(s,t) = \begin{cases} H(2s,t), & s \in [0, \frac{1}{2}] \\ G(2s-1,t), & s \in [\frac{1}{2}, 1] \end{cases}$

Доказ: * непрекидност

$[0, \frac{1}{2}] \times I \qquad [\frac{1}{2}, 1] \times I$

$\underbrace{F|_{[0,\frac{1}{2}] \times I}}_{= H \circ \alpha} : [0, \frac{1}{2}] \times I \xrightarrow{\alpha} I \times I \xrightarrow{H} X$

$\alpha(s,t) = (2s, t)$

$F|_{[0,\frac{1}{2}] \times I}$ непр. као комп. непрекидних

аналогно $F|_{[\frac{1}{2},1] \times I}$ непрекидно

лема о лепљењу $\Rightarrow F$ непрекидно ✓

* добро деф. $s = \frac{1}{2}, \ t \in I$

$H(1,t) = x_0 = G(0,t)$ ✓

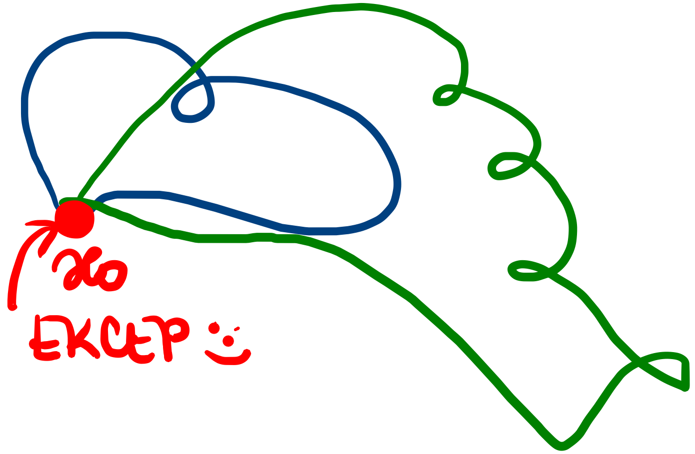
Хомотопни патишта, rel$\{0,1\}$

Став $u, u', v, v' \in \mathcal{P}(X, x_0)$ $\quad (u, u', v, v' : I \to X)$

$u \simeq u' \ (\mathrm{rel}\{0,1\})$

$v \simeq v' \ (\mathrm{rel}\{0,1\})$

$$\Rightarrow \quad u \cdot v \simeq u' \cdot v' \ (\mathrm{rel}\{0,1\})$$

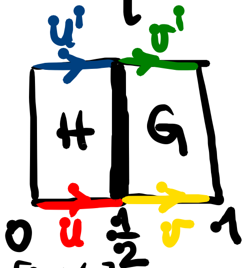


Доказ: Потребно да

$F: u \cdot v \simeq u' \cdot v' \ (\mathrm{rel}\{0,1\})$

$$u \cdot v(s) = \begin{cases} u(2s), & s \in [0, \frac{1}{2}] \\ v(2s-1), & s \in [\frac{1}{2}, 1] \end{cases}$$

Конструирано: $F: I \times I \to X$



$$F(s,t) = \begin{cases} H(2s, t), & s \in [0, \frac{1}{2}] \\ G(2s-1, t), & s \in [\frac{1}{2}, 1] \end{cases}$$

* добро деф. $s = \frac{1}{2}$, $t \in I$

$H(1,t) = x_0 = G(0,t)$ ✓

- $F(s,0) = \begin{cases} H(2s,0), & s \in [0,\frac{1}{2}] \\ G(2s-1,0), & s \in [\frac{1}{2},1] \end{cases} = \begin{cases} u(2s), & s \in [0,\frac{1}{2}] \\ v(2s-1), & s \in [\frac{1}{2},1] \end{cases}$

$\boxed{F(s,0) = (u \cdot v)(s), \ \forall s \in I}$

- аналогно: $\boxed{F(s,1) = (u' \cdot v')(s), \ \forall s \in I}$

- $F(0,t) = H(0,t) = x_0$
  $F(1,t) = G(1,t) = x_0$ $\quad \forall t \in I$

$$F: u \cdot v \simeq u' \cdot v' \ (\mathrm{rel}\{0,1\}) \ \square$$

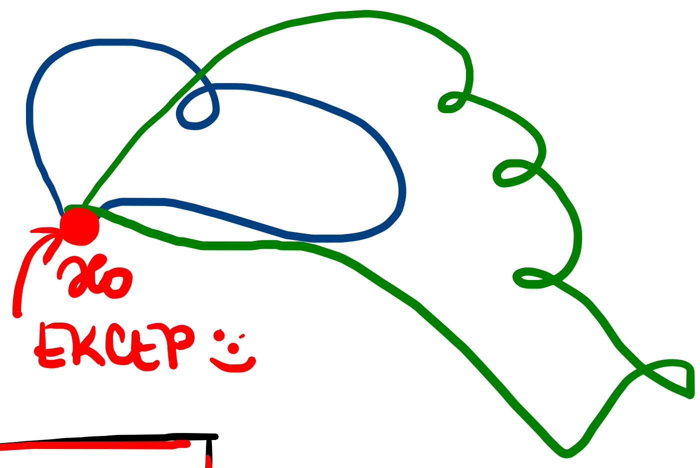
Хомотопне петље, $\mathrm{rel}\{0,1\}$

Став $u, u', v, v' \in P(X, x_0)$ $\quad (u, u', v, v' : I \to X)$

$u \simeq u' \ (\mathrm{rel}\{0,1\})$

$v \simeq v' \ (\mathrm{rel}\{0,1\})$

$\Rightarrow \ u \cdot v \simeq u' \cdot v' \ (\mathrm{rel}\{0,1\})$



$P(X, x_0)$ $\simeq (\mathrm{rel}\{0,1\}) \to$ рел. еквиваленције

количнички скуп: $P(X, x_0) / \simeq_{(\mathrm{rel}\{0,1\})} =: \boxed{\pi_1(X, x_0)}$

$\pi_1(X, x_0) \ni [u]$ – класа петље

$[u] = [u'] \Leftrightarrow u \simeq u' \ (\mathrm{rel}\{0,1\})$

Правимо структуру групе:

① ОПЕРАЦИЈА $*$

$[u], [v] \in \pi_1(X, x_0)$

$[u] * [v] := [u \cdot v]$

$* : \pi_1(X, x_0) \times \pi_1(X, x_0) \to \pi_1(X, x_0)$

добро деф. - из претх. става (не зависи од предс. класе)

$[u] = [u']$

$[v] = [v']$

$[u] * [v] = [u \cdot v] \overset{\text{став}}{=} [u' \cdot v'] = [u'] * [v']$

✓

② неутрал

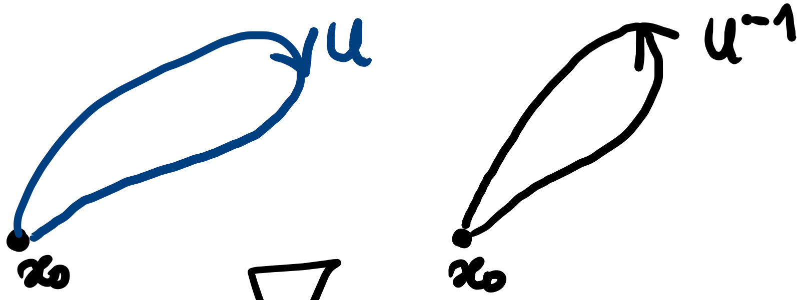
$c_{x_0}: I \to X \quad \forall t \in I \quad c_{x_0}(t) := x_0$

③ за инверзе (кандидати)

$u: I \to X$ петља у x_0

петља у супр. смеру: $u^{-1}: I \to X$

$u^{-1}(t) := u(1-t)$



! u^{-1} није инверзно за u !

$[u^{-1}]$ ће бити инверз у групи $[u]$

$\boxed{\text{Теорема}}$ $(\pi_1(X,x_0), *)$ је група

Њен неутрал је $[c_{x_0}]$, а за $\forall [u] \in \pi_1(X,x_0)$ инверзни елемент је $[u]^{-1} = [u^{-1}]$.

Доказ: I асоц, II неутрал, III инверз

I Асоцијативност

$u, v, w \in P(X,x_0)$

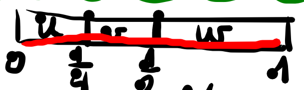
$$([u]*[v])*[w] \overset{?}{=} [u]*([v]*[w])$$

$$(u \cdot v) \cdot w \simeq u \cdot (v \cdot w) \quad \text{rel}\{0,1\}$$

?

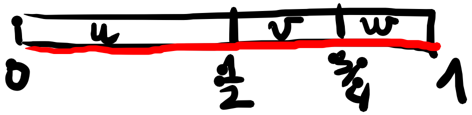
$(u\cdot v)\cdot w \underset{?}{\simeq} u\cdot(v\cdot w)\ (\mathrm{rel}\{0,1\})$? хомотопия F

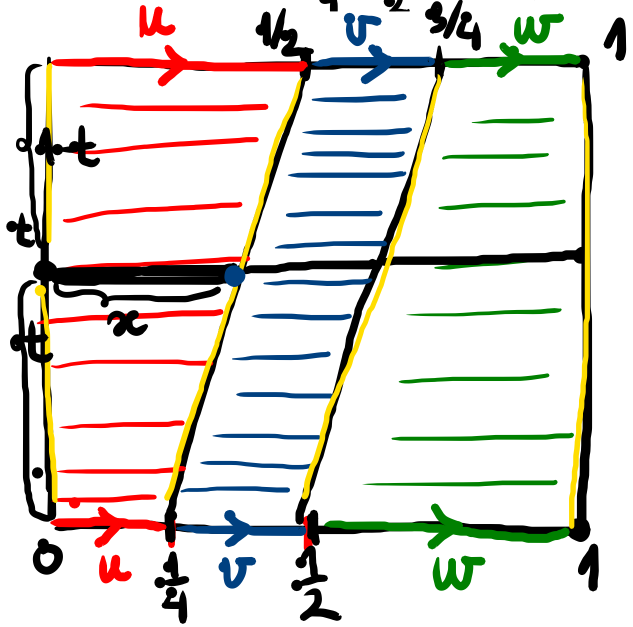
$s\in I:\ (u\cdot v)\cdot w(s) = \begin{cases}(u\cdot v)(2s), & s\in[0,\frac12]\\ w(2s-1), & s\in[\frac12,1]\end{cases} = \begin{cases}u(4s), & s\in[0,\frac14]\\ v(4s-1), & s\in[\frac14,\frac12]\\ w(2s-1), & s\in[\frac12,1]\end{cases}$



Аналогично:

$u\cdot(v\cdot w)(s) = \begin{cases} u(2s), & s\in[0,\frac12]\\ v(4s-2), & s\in[\frac12,\frac34]\\ w(4s-3), & s\in[\frac34,1]\end{cases}$

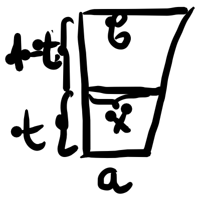




$\boxed{F: I\times I\to X}$

de / → x_0

1 сл



$x=(1-t)\cdot a+t\cdot b$

за нас $a=\frac14,\ b=\frac12$

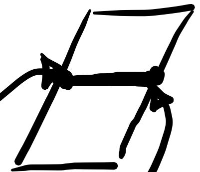
$x=(1-t)\cdot\frac14+t\cdot\frac12=\frac{t+1}{4}$

$\Rightarrow$ за $s\in[0,\frac{t+1}{4}]$

$F(s,t):=u\left(\frac{s}{\frac{t+1}{4}}\right)$

$F(s,t)=u\left(\frac{4s}{t+1}\right)$ (*)

2 сл



за $s\in[\frac{t+1}{4},\frac{t+2}{4}]$

$F(s,t):=v\left(\frac{s-\frac{t+1}{4}}{1/4}\right)$

$\frac{t+1}{4}$

$\frac{t+1}{4}+\frac14=\frac{t+2}{4}$

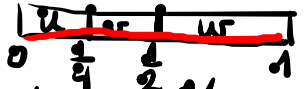
$F(s,t)=v(4s-t-1)$ (**)

3 сл $s\in[\frac{t+2}{4},1]\to$ дължина сегмента $=1-\frac{t+2}{4}=\frac{2-t}{4}$

$F(s,t):=w\left(\frac{s-\frac{t+2}{4}}{\frac{2-t}{4}}\right)=w\left(1-\frac{4(1-s)}{2-t}\right)$ (***)

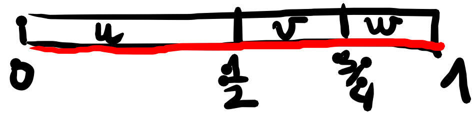
$(u\cdot v)\cdot w \overset{?}{\simeq} u\cdot(v\cdot w)\ (\mathrm{rel}\{0,1\})$? хомотопија F

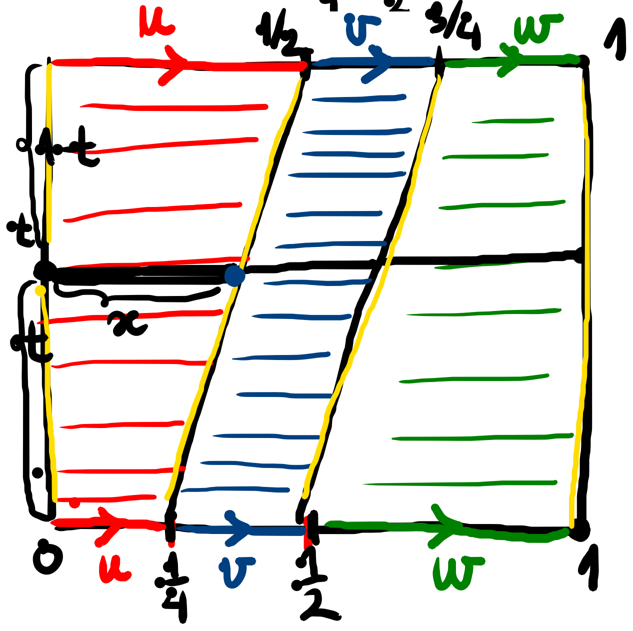
$s\in I$: $\quad (u\cdot v)\cdot w(s) = \begin{cases} (u\cdot v)(2s), & s\in[0,\frac{1}{2}] \\ w(2s-1), & s\in[\frac{1}{2},1] \end{cases} = \begin{cases} u(4s), & s\in[0,\frac{1}{4}] \\ v(4s-1), & s\in[\frac{1}{4},\frac{1}{2}] \\ w(2s-1), & s\in[\frac{1}{2},1] \end{cases}$



Аналогно:

$u\cdot(v\cdot w)(s) = \begin{cases} u(2s), & s\in[0,\frac{1}{2}] \\ v(4s-2), & s\in[\frac{1}{2},\frac{3}{4}] \\ w(4s-3), & s\in[\frac{3}{4},1] \end{cases}$





$\boxed{F: I\times I \to X}$ (*) (**) (***)

$F(s,t) = \begin{cases} u\left(\frac{4s}{1+t}\right), & s\in[0,\frac{t+1}{4}] \\ v(4s-1-t), & s\in[\frac{t+1}{4},\frac{t+2}{4}] \\ w\left(1-\frac{4(1-s)}{2-t}\right), & s\in[\frac{t+2}{4},1] \end{cases}$

- Непрекидно? ✓ (лема о лепљењу ++)

$\boxed{t=0}$ $\quad F(s,0) = \begin{cases} u(4s), & s\in[0,\frac{1}{4}] \\ v(4s-1), & s\in[\frac{1}{4},\frac{1}{2}] \\ w(2s-1), & s\in[\frac{1}{2},1] \end{cases} = (u\cdot v)\cdot w(s)$ ✓

$\boxed{t=1}$ $\quad F(s,1) \overset{\dots}{=} u\cdot(v\cdot w)(s)$

↑ аналогно

$\boxed{\begin{matrix} s=0 \\ s=1 \end{matrix}}$ $\quad F(0,t)=u(0)=x_0$ ✓ $\quad F(1,t)=w(1)=x_0$ ✓

$\Rightarrow F: (u\cdot v)\cdot w \simeq u\cdot(v\cdot w)\ (\mathrm{rel}\{0,1\})$

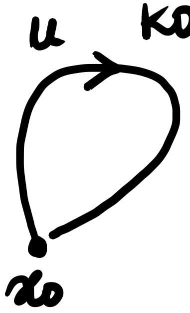
$\Rightarrow$ $*$ је АСОЦИЈАТИВНА

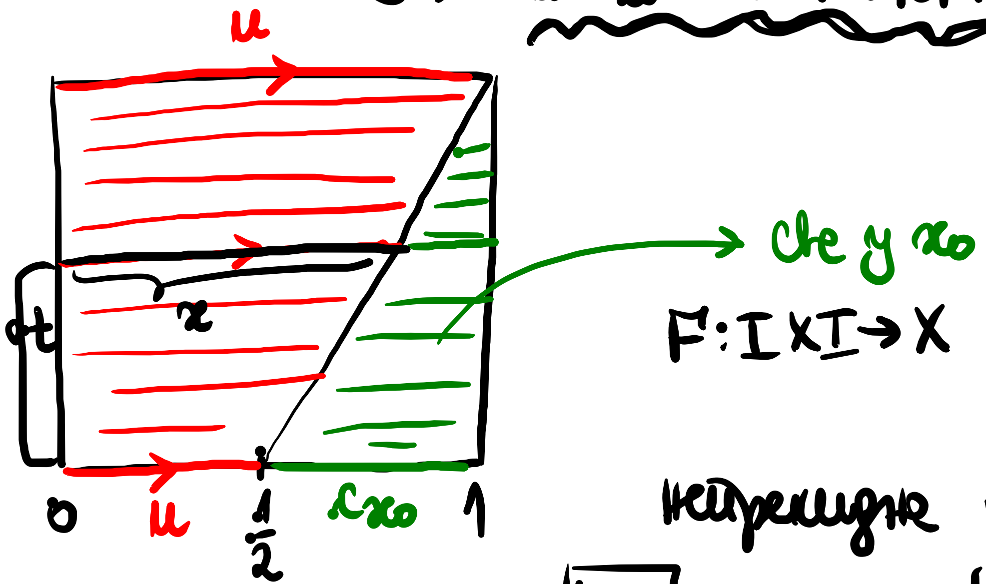
II НЕУТРАЛ: $[c_{x_0}]$ – неутрал у $\pi_1(X, x_0)$?

$u \in P(X, x_0)$ желимо: $[u] * [c_{x_0}] = [u]$, $[c_{x_0}] * [u] = [u]$?

$\Longleftrightarrow$ $\underline{u \cdot c_{x_0} \simeq u \;(\mathrm{rel}\{0,1\})}$, $c_{x_0} \cdot u \simeq u \;(\mathrm{rel}\{0,1\})$

! не мора бити конс !





$x = t \cdot 1 + (1-t) \cdot \frac{1}{2} = \frac{t+1}{2}$

$s \in [0, \frac{t+1}{2}]$ $\quad \frac{s}{\frac{t+1}{2}} = \frac{2s}{t+1}$

$F : I \times I \to X$

$$F(s,t) = \begin{cases} u\left(\frac{2s}{t+1}\right), & s \in [0, \frac{t+1}{2}] \\ x_0, & s \in [\frac{t+1}{2}, 1] \end{cases}$$

непрекидне ✓

$\boxed{t=0}$ $\quad F(s,0) = \begin{cases} u(2s), & s \in [0, \frac{1}{2}] \\ x_0, & s \in [\frac{1}{2}, 1] \end{cases} = u \cdot c_{x_0}(s)$

$\boxed{t=1}$ $\quad F(s,1) = \begin{cases} u(s), & s \in [0, 1] \\ x_0, & s = 1 \end{cases} = u(s)$

$F(0,t) = u(0) = x_0$ ✓ $\quad F(1,t) = x_0$ ✓

$F : u \cdot c_{x_0} \simeq u$ $(\mathrm{rel}\{0,1\})$

аналогно c_{x_0} са друге стр.

$\Rightarrow [c_{x_0}]$ је НЕУТРАЛ

III ИНВЕРЗ $u \in P(X; x_0)$ $u^{-1}(t) := u(1-t)$

Твърдим: $[u] * [u^{-1}] = [c_{x_0}] = [u^{-1}] * [u]$

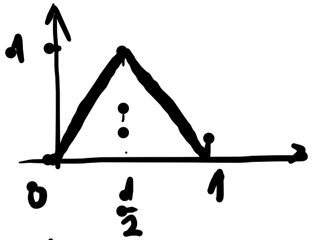
Д-во. $u \cdot u^{-1} \simeq c_{x_0} \, (\mathrm{rel}\{0,1\})$ (1)

$u^{-1} \cdot u \simeq c_{x_0} \, (\mathrm{rel}\{0,1\})$ (2)

Забележка: $(u^{-1})^{-1}(t) = u^{-1}(1-t) = u(1-(1-t)) = u(t) \to (u^{-1})^{-1} = u \to$ достатъчно док. само (1)

(1)! Нека: $g : I \to I$

$$g(t) = \begin{cases} 2t, & 0 \le t \le \frac{1}{2} \\ 2-2t, & \frac{1}{2} \le t \le 1 \end{cases}$$



g непрекъсната, $g(0) = g(1) = 0$

$g : I \to I$ конвексно $\Rightarrow$ $g \simeq 0 \, (\mathrm{rel}\{0,1\})$

където $\forall t \in I \; 0(t) = 0$

следв.

$\Rightarrow$ $\boxed{u \circ g \simeq u \circ 0 \, (\mathrm{rel}\{0,1\})}$

дясно: $u \circ 0(t) = u(0) = x_0$ $u \circ 0 = c_{x_0}$

ляво: $$u \circ g(t) = \begin{cases} u(2t), & t \in [0, \frac{1}{2}] \\ u(2-2t), & t \in [\frac{1}{2}, 1] \end{cases} = \begin{cases} u(2t), & t \in [0, \frac{1}{2}] \\ \underbrace{u(1-(2t-1))}_{u^{-1}(2t-1)}, & t \in [\frac{1}{2}, 1] \end{cases} = (u \cdot u^{-1})(t)$$

$u \cdot u^{-1} \simeq c_{x_0} (\mathrm{rel}\{0,1\})$ ✓✓

□

Деф $\pi_1(X,x_0)$ – фундаментална група простора X са базном тачком x_0.
(са $*$)

Тв. $X=\{*\}$ $\quad \pi_1(*)=0$

Својства фундаменталне групе

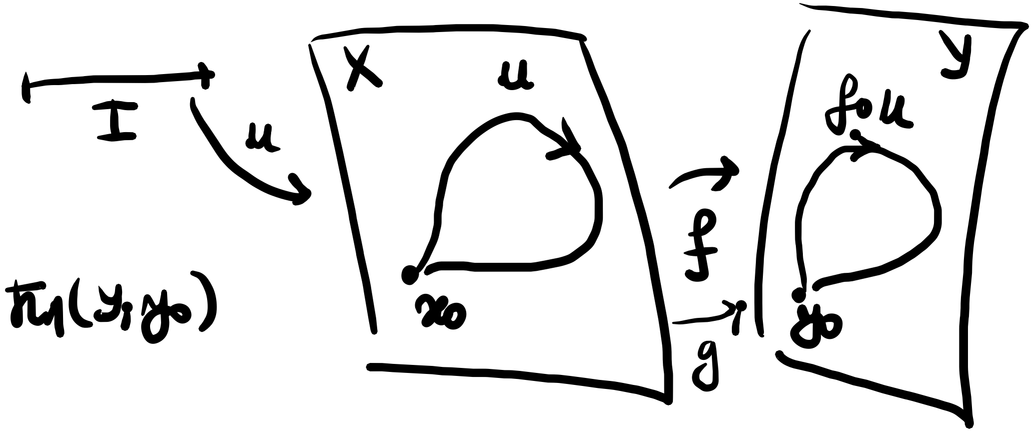
$\pi_1(X,x_0)$ & функције: $(X,x_0), (Y,y_0)$

непр. $f: X\to Y$ $\quad f(x_0)=y_0$

f индукује $f_*: \pi_1(X,x_0)\to\pi_1(Y,y_0)$

$[u]\in\pi_1(X,x_0)$ $\quad u: I\to X$

$$f_*[u] := [f\circ u]$$



f_* је хомоморфизам група

- $f_*[c_{x_0}]=[c_{y_0}]$
- $f_*([u]*[v]) = f_*[u]*f_*[v]$

- $(1_X)_* = 1_{\pi_1(X,x_0)}$
- $(g\circ f)_* = g_*\circ f_*$
- $\left.\begin{array}{l} f,g:(X,x_0)\to(Y,y_0) \\ f\simeq g \ \mathrm{rel}\{x_0\}\end{array}\right\} \Rightarrow f_*=g_*$

• Занемаривање базне тачке –

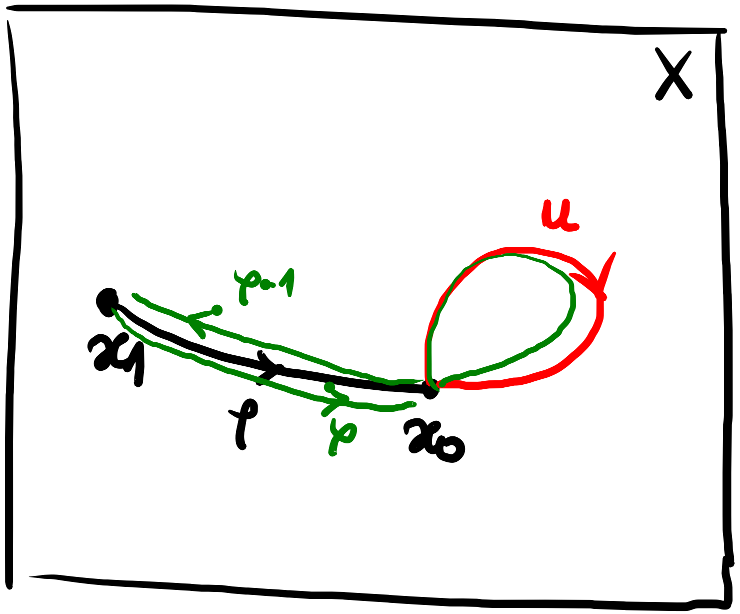
<u>идеја:</u> $x_0, x_1 \in X$

$\exists$ пут φ од x_1 до x_0

$$[u] \in \pi_1(X, \underline{\underline{x_0}})$$

$$\Big\downarrow \beta_\varphi$$

$$[\varphi \cdot u \cdot \varphi^{-1}] \in \pi_1(X, \underline{\underline{x_1}})$$



<u>Теорема</u> β_φ је изоморфизам група.

<u>Последица</u> X путно повезан $\Rightarrow (\forall x_0, x_1 \in X)\ \pi_1(X, x_0) \cong \pi_1(X, x_1)$

$\rightsquigarrow$ Говоримо само о $\pi_1(X)$

фундаментална група простора X