~ Смена променљиве у двоструком интегралу ~

$\boxed{T}$ D, D_1 - отв., повезани, мерљиви у $\mathbb{R}^2$; $F: \overline{D_1} \to \overline{D}$ непр-диф., бij: $D_1 \to D$ $\Big\} \Rightarrow \iint\limits_D f(x,y)\,dx\,dy = \iint\limits_{D_1} f(F(u,v)) \cdot |J_F(u,v)|\,du\,dv$

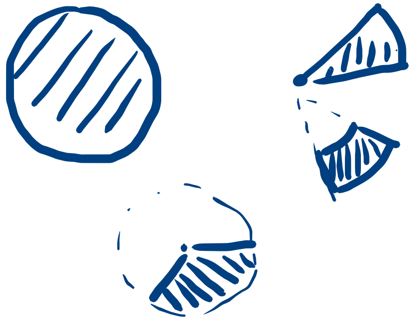
$$\iint\limits_D f(x,y)\,dx\,dy \rightsquigarrow \begin{array}{l} x = F_1(u,v) \\ y = F_2(u,v) \end{array}$$

$$F(u,v) = (F_1(u,v), F_2(u,v)) = (x,y)$$

$F: \sqcup \to \sqcup$ услови ... ✓

$$J_F(u,v) = \det \begin{bmatrix} x'_u & x'_v \\ y'_u & y'_v \end{bmatrix}$$

ПОЛАРНА СМЕНА:

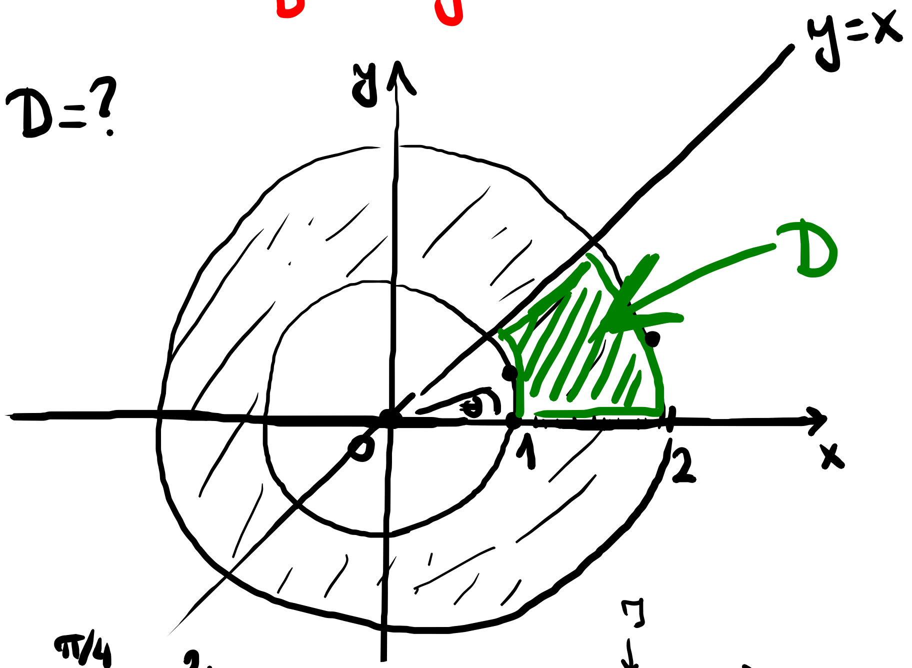


$$\left.\begin{array}{l} x = r \cdot \cos\theta \\ y = r \cdot \sin\theta \end{array}\right\} \quad |J_F(r,\theta)| = r$$

$\rightsquigarrow$ $r \in ?$
$\theta \in ?$

1. $I = \iint\limits_D \frac{1}{2x^2+y^2}\,dx\,dy \qquad D = \{(x,y)\in\mathbb{R}^2 \mid \underbrace{1\le x^2+y^2\le 4}_{\text{кружни прстен између кругова}},\ 0\le y\le x\}$

D=?



$$\boxed{x = r\cdot\cos\theta,\ y = r\cdot\sin\theta} \qquad \boxed{J = r}$$

- $1\le \underbrace{x^2+y^2}_{r^2}\le 4 \Rightarrow 1\le r\le 2,\ r\ge 0 \qquad \boxed{r\in[1,2]}$
- $\theta\in\,? \qquad \boxed{\theta\in[0,\frac{\pi}{4}]}$

$$I = \int_0^{\pi/4}\left(\int_1^2 \frac{1}{2r^2\cos^2\theta + r^2\sin^2\theta}\cdot \overset{J}{r}\,dr\right)d\theta = \int_0^{\pi/4}\left(\int_1^2 \frac{1}{r}\cdot\frac{1}{\underbrace{2\cos^2\theta+\sin^2\theta}_{1+\cos^2\theta}}\,dr\right)d\theta = \int_0^{\pi/4}\frac{1}{1+\cos^2\theta}\cdot \underbrace{\ln r\Big|_1^2}_{\ln 2-\ln 1 = \ln 2}\,d\theta$$

$$I = \ln 2\cdot\int_0^{\pi/4}\frac{1}{1+\cos^2\theta}\,d\theta \overset{A}{=} \ln 2\cdot\int_0^1 \frac{1}{1+\frac{1}{1+t^2}}\cdot\frac{dt}{1+t^2} = \ln 2\cdot\int_0^1\frac{dt}{2+t^2}$$

Смена: $t = \mathrm{tg}\,\theta \rightsquigarrow \cos^2\theta = \frac{1}{1+t^2}$

$\theta\in[0,\frac{\pi}{4}] \to t\in[0,1] \qquad d\theta = \frac{1}{1+t^2}\,dt$

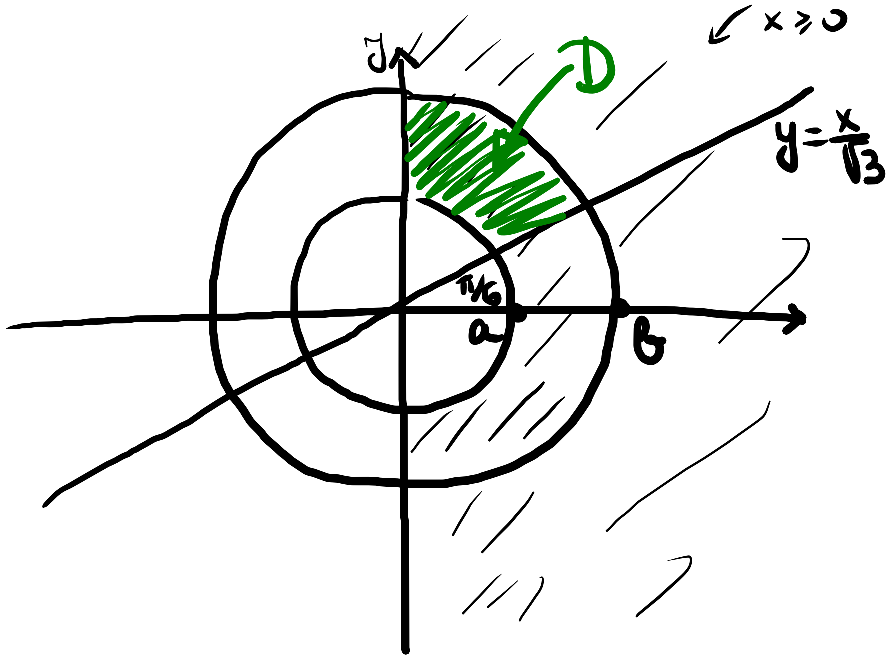
$$= \ln 2\cdot\frac{1}{\sqrt2}\cdot\mathrm{arctg}\,\frac{t}{\sqrt2}\Big|_0^1$$

$$= \boxed{\ln 2\cdot\frac{1}{\sqrt2}\cdot\mathrm{arctg}\,\frac{1}{\sqrt2}} \qquad \square$$

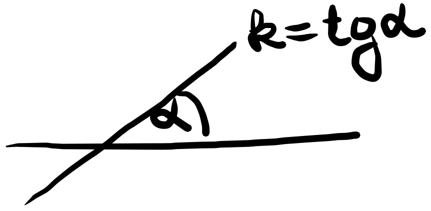
За вежбу:

$$I=\iint_D e^{x^2+y^2}\,dx\,dy$$

$$D=\left\{(x,y)\in\mathbb{R}^2 \mid a^2\le x^2+y^2\le b^2,\ x\ge 0,\ y\ge \frac{x}{\sqrt{3}}\right\} \qquad a,b>0,\ a<b$$



$y=\frac{1}{\sqrt{3}}\cdot x$ - права



$\frac{1}{\sqrt{3}}=\operatorname{tg}\alpha$

$$\boxed{\alpha=\frac{\pi}{6}}$$

$x=r\cos\theta$

$y=r\sin\theta$

$$\boxed{\begin{aligned} &r\in[a,b]\\ &\theta\in\left[\frac{\pi}{6},\frac{\pi}{2}\right]\end{aligned}}$$

!! $\int e^{t^2}t\,dt$

смена $z=t^2$

② $I = \iint\limits_D (x+y)\,dx\,dy \qquad D = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 \le x+y\}$

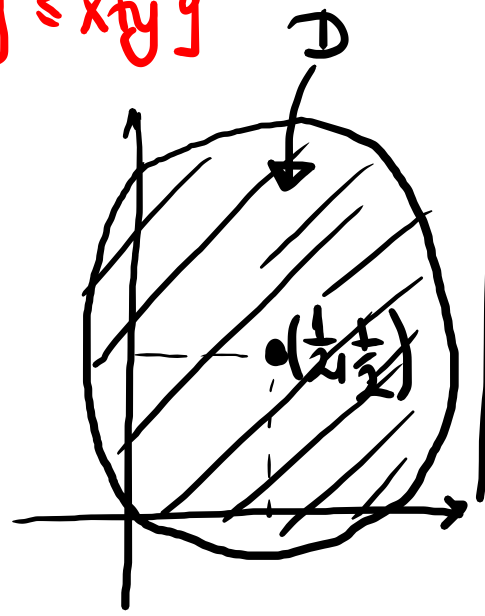
$D:\ x^2+y^2 \le x+y$

$\Leftrightarrow x^2 - x + y^2 - y \le 0$

$\Leftrightarrow \boxed{x^2 - 2\cdot\frac{1}{2}x + \frac{1}{4}} - \frac{1}{4} + \boxed{y^2 - 2\frac{1}{2}y + \frac{1}{4}} - \frac{1}{4} \le 0$

$\Leftrightarrow \left(x-\frac{1}{2}\right)^2 + \left(y-\frac{1}{2}\right)^2 \le \frac{1}{2}$

$D =$ круг со центар $\left(\frac{1}{2},\frac{1}{2}\right)$, радиус $= \frac{1}{\sqrt{2}}$



ТРАНСЛИРАМЕ ПОЛАРНИ КООРД.

$$\boxed{\begin{aligned} x &= r\cos\theta + \frac{1}{2} \\ y &= r\sin\theta + \frac{1}{2} \end{aligned}}$$

$$\boxed{\begin{aligned} r &\in \left[0, \frac{1}{\sqrt{2}}\right] \\ \theta &\in [-\pi, \pi) \end{aligned}} \quad [0, 2\pi)$$

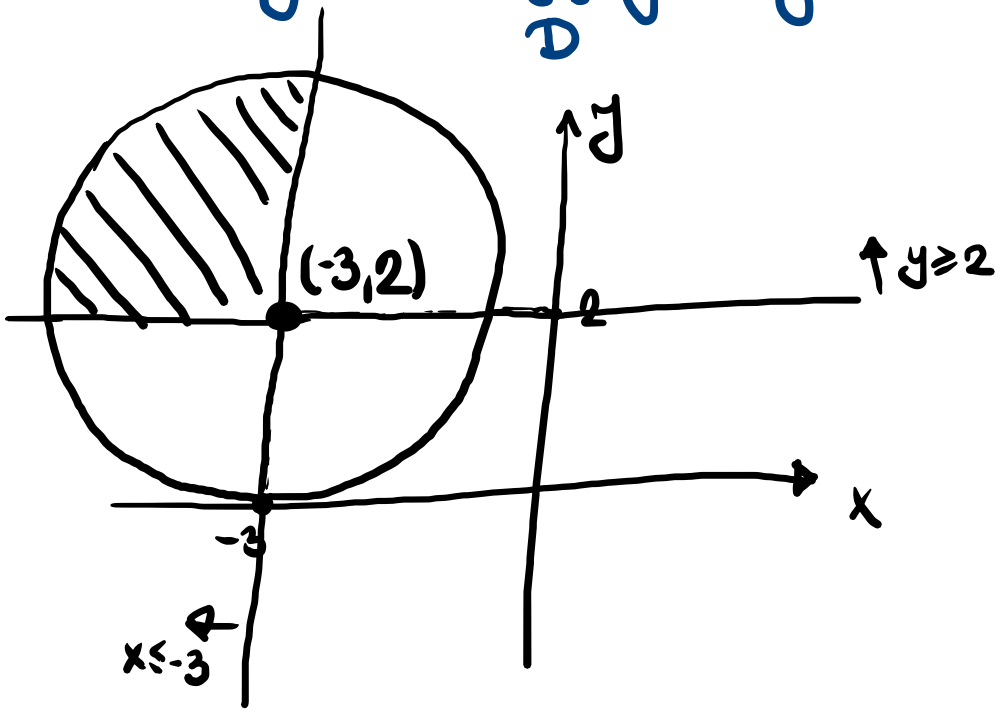
$$J = \det\begin{bmatrix} x'_r & x'_\theta \\ y'_r & y'_\theta \end{bmatrix} = \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix} = r\cdot(\cos^2\theta) - -r\sin^2\theta = \boxed{r}$$

$$I = \int_{-\pi}^{\pi}\left(\int_0^{1/\sqrt{2}} \underbrace{\left(r\cos\theta + \frac{1}{2} + r\sin\theta + \frac{1}{2}\right)}_{x+y} \cdot r\,dr\right)d\theta = \int_{-\pi}^{\pi}\left(\int_0^{1/\sqrt{2}} \left(r + r^2(\sin\theta + \cos\theta)\right)dr\right)d\theta$$

$$= \int_{-\pi}^{\pi} \left(\frac{r^2}{2} + \frac{r^3}{3}(\sin\theta + \cos\theta)\right)\Bigg|_0^{1/\sqrt{2}} d\theta = \ldots = \boxed{\frac{\pi}{2}} \quad ■$$

<u>за вежбу</u>: $I=\iint\limits_D y\,dx\,dy$ $\quad D=\{(x,y)\in\mathbb{R}^2 \mid \underbrace{x^2+y^2+6x-4y+9\le 0}_{(x+3)^2+(y-2)^2\le 2^2},\ y\ge 2,\ x\le -3\}$

центар $(-3,2)$



$x = r\cos\theta - 3$ $\qquad J = r$

$y = r\sin\theta + 2$

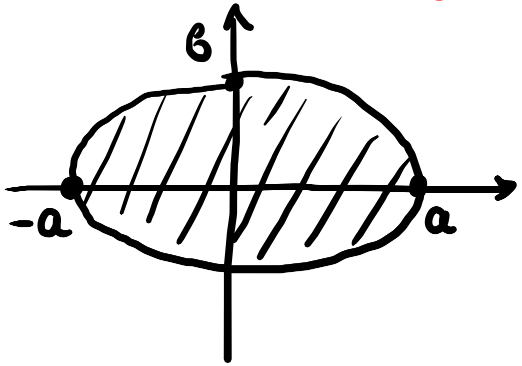
$r\in[0,2]$

$\theta\in\left[\frac{\pi}{2},\pi\right]$

$$\rightsquigarrow \int_{\frac{\pi}{2}}^{\pi}\left(\int_0^2 (r\sin\theta+2)\cdot r\,dr\right)d\theta = \ldots$$

3. Израчунати $P(\mathcal{E})$ $\quad \mathcal{E}: \frac{x^2}{a^2}+\frac{y^2}{b^2}\le 1,\ a,b>0$

$$P(\mathcal{E}) = \iint_{\mathcal{E}} dx\,dy$$



I начин:

$$\int_{-a}^{a}\left(\int_{-\sqrt{\dots}}^{+\sqrt{\dots}} 1\,dy\right)dx = P(\mathcal{E})$$

II начин:

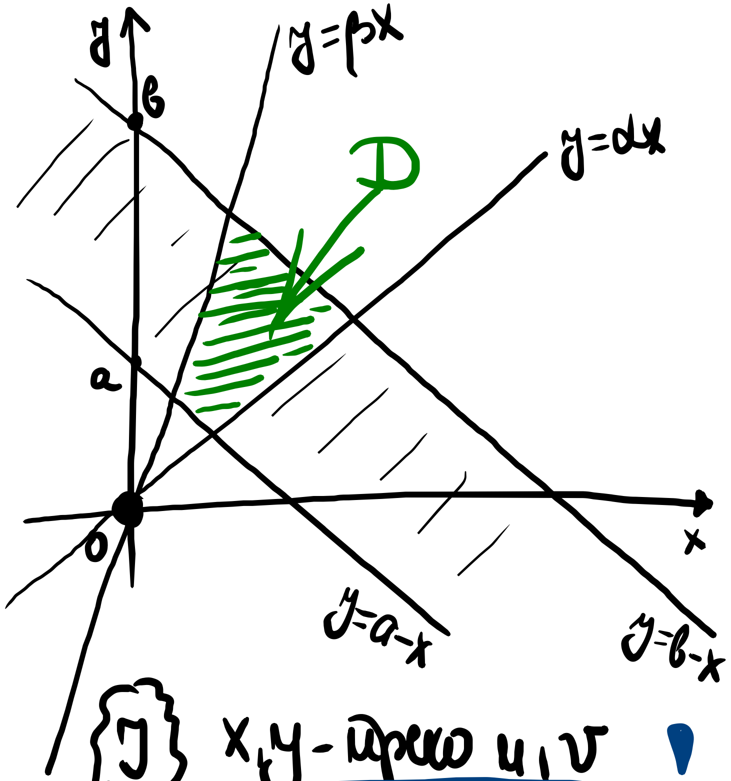
$$\boxed{\begin{array}{ll} x = a\cdot r\cos\theta & r\in[0,1] \\ y = b\cdot r\sin\theta & \theta\in[0,2\pi) \end{array}}$$

$$\boxed{J} = \det\begin{bmatrix} x'_r & x'_\theta \\ y'_r & y'_\theta \end{bmatrix} = \det\begin{bmatrix} a\cos\theta & -ar\sin\theta \\ b\sin\theta & br\cos\theta \end{bmatrix} = abr(\cos^2\theta+\sin^2\theta) = \boxed{abr}$$

$$P(\mathcal{E}) = \iint_{\mathcal{E}} dx\,dy = \int_0^1\left(\int_0^{2\pi} 1\cdot abr\,d\theta\right)dr = \int_0^1 abr\cdot\underbrace{\theta\Big|_0^{2\pi}}_{2\pi}dr = 2\pi ab\underbrace{\int_0^1 r\,dr}_{r^2/2|_0^1=\frac{1}{2}} = 2\pi ab\cdot\frac{1}{2} = \underline{\underline{ab\pi}}$$

$$\boxed{P(\mathcal{E}) = ab\pi}$$

4. $I=\iint\limits_D \frac{1}{xy}\,dxdy \quad D=\{(x,y)\in\mathbb{R}^2 \mid a\le x+y\le b,\ \underbrace{\alpha x\le y\le\beta x}_{\text{између}}\} \quad 0<a<b,\ 0<\alpha<\beta$



$a=x+y \quad y=a-x$

$b=x+y \quad y=b-x$

између

$y=\alpha x$

$y=\beta x$

$\boxed{a\le x+y\le b}$

$\alpha x\le y\le \beta x \ \ /:x>0$ на D $\rightarrow \boxed{\alpha\le \frac{y}{x}\le\beta}$

смена: $\boxed{\begin{array}{ll} u=x+y, & u\in[a,b] \\ v=\frac{y}{x}, & v\in[\alpha,\beta]\end{array}}$

J: x, y - преко u, v !

$u=x+y \Rightarrow x(v+1)=u$

$vx=y \Rightarrow \boxed{x=\frac{u}{v+1}}$

$\boxed{y=\frac{uv}{v+1}}$

$J_F=\det\begin{bmatrix} x'_u & x'_v \\ y'_u & y'_v\end{bmatrix}=\det\begin{bmatrix}\frac{1}{v+1} & -\frac{u}{(v+1)^2} \\ \frac{v}{v+1} & \frac{u}{(v+1)^2}\end{bmatrix} \overset{\dots}{=} \frac{u}{(v+1)^2}$

$|J_F| \underset{u>0}{=} J=\frac{u}{(v+1)^2}$

$$I = \iint_D \frac{1}{xy}\,dx\,dy \overset{\text{смена}}{=\!=} \int_a^b \int_\alpha^\beta \frac{1}{\frac{u}{v+1}\cdot\frac{uv}{v+1}}\cdot\frac{u}{(v+1)^2}\,dv\,du = \int_a^b\int_\alpha^\beta \frac{1}{uv}\,du\,dv \overset{\dots}{=} \ln\frac{\beta}{\alpha}\cdot\ln\frac{b}{a}$$

За вежбу:

$$I = \iint_D \frac{x^2+y^2}{xy}\,dx\,dy$$

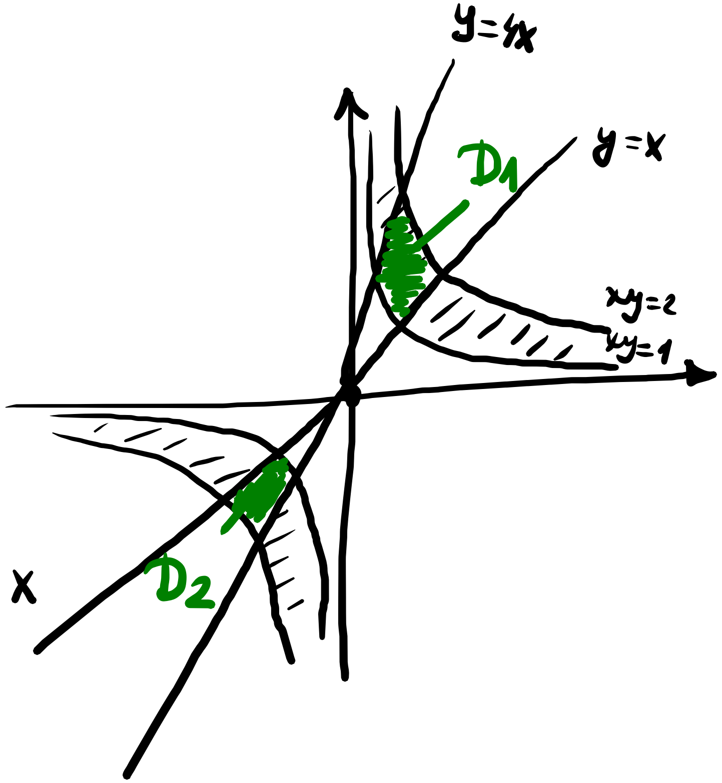
$D \subset \mathbb{R}^2$: $\quad 1 \le xy \le 2$, $\quad y^2 - 5xy + 4x^2 \le 0 \;/ : x^2$

$xy = u$

$(y-4x)(y-x) \le 0$

$x \le y \le 4x \;\lor\; 4x \le y \le x$

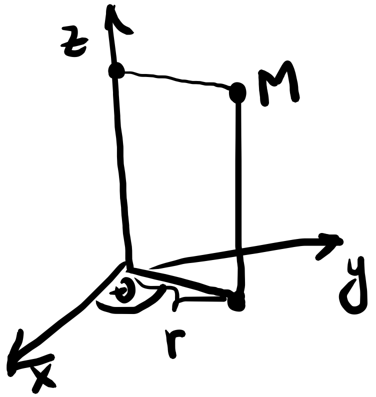
$v = \frac{y}{x}$



~ Смена променљиве у троструком интегралу ~

[T] ... $\iiint\limits_{T} f(x,y,z)\,dx\,dy\,dz = \iiint\limits_{T_1} f(F(u,v,w))\cdot |J_F(u,v,w)|\,du\,dv\,dw$

ЦИЛИНДРИЧНЕ КООРДИНАТЕ



$M(x,y,z) \to M(r,\theta,z)$

$x = r\cos\theta$

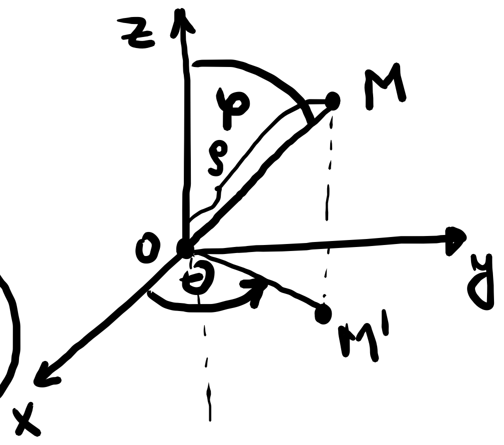
$y = r\sin\theta$

$r \geq 0,\ \theta \in [0,2\pi),\ z \in \mathbb{R}$

$\boxed{M(r\cos\theta, r\sin\theta, z)}$

$\boxed{J = r}$

СФЕРНЕ КООРДИНАТЕ



$M(x,y,z) \to M(\rho,\theta,\varphi)$

$\rho \geq 0$

$\theta \in [0,2\pi)\ /\ [-\pi,\pi)$

$\varphi \in [0,\pi]$

$$\boxed{\begin{aligned} x &= \rho\sin\varphi\cos\theta \\ y &= \rho\sin\varphi\sin\theta \\ z &= \rho\cos\varphi \end{aligned}}$$

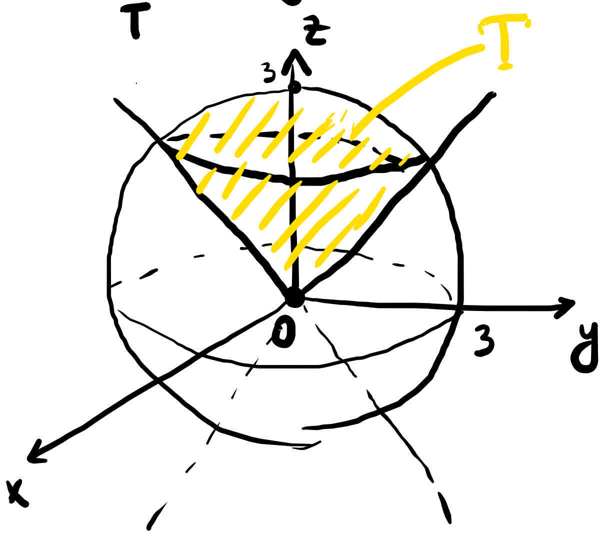
$J = -\rho^2 \sin\varphi \leq 0$

$|J| = \rho^2 \sin\varphi$

1. $V(T) = ?$ $\quad T = \{(x,y,z) \in \mathbb{R}^3 \mid \underbrace{x^2+y^2+z^2 \le 9}_{\text{унутр. сфере}},\ \underbrace{z^2 \ge \frac{x^2+y^2}{3}}_{\text{конус}},\ z \ge 0\}$

$$V(T) = \iiint_T dx\,dy\,dz$$

T:



I начин: фуб. теорема

пр(T) = D

$$\iint_D \Big(\int_{z=\dots \text{ из конуса}}^{z=\dots \text{ из сфере}} 1\,dz \Big) dx\,dy$$

II начин:

$$\boxed{\begin{aligned} x &= \rho \cdot \sin\varphi \cdot \cos\theta \\ y &= \rho \cdot \sin\varphi \cdot \sin\theta \\ z &= \rho \cdot \cos\varphi \\ |J| &= \rho^2 \sin\varphi \end{aligned}}$$

• $\boxed{\rho \in [0,3]}$

$(9 \ge x^2+y^2+z^2 = \rho^2 \Rightarrow \rho \le 3)$

• $\theta \in ?$

$\boxed{\theta \in [-\pi, \pi)}$

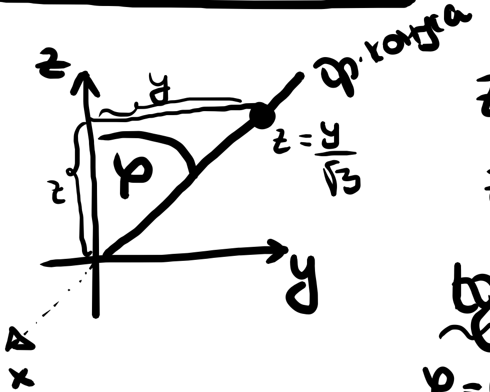
(сви углови у xy-равни)

• $\varphi \in ?$ $\quad \varphi \in [0,$ "граница конуса"$]$

min

у yz-равни $\boxed{x=0}$

$$z^2 = \frac{x^2+y^2}{3} \Rightarrow \frac{y^2}{3}$$



$z^2 = \frac{y^2}{3}$ $\quad z, y > 0$

$z = \frac{y}{\sqrt{3}}$

$\operatorname{tg}\varphi = \frac{y}{z} = \sqrt{3}$

$\varphi = \operatorname{arctg}\sqrt{3}$ $\quad \varphi = \frac{\pi}{3}$ max

$$\boxed{\varphi \in \left[0, \frac{\pi}{3}\right]}$$

$$V(T) = \iiint_T dx\,dy\,dz \overset{\text{смена}}{=} \underset{\theta}{\int_{-\pi}^{\pi}} \underset{\varphi}{\int_{0}^{\pi/3}} \underset{\rho}{\int_{0}^{3}} \underbrace{\rho^2 \cdot \sin\varphi}_{|J|} \, d\rho\, d\varphi\, d\theta$$

$$= \int_{-\pi}^{\pi} \int_{0}^{\pi/3} \sin\varphi \cdot \frac{\rho^3}{3}\Big|_0^3 \, d\varphi\, d\theta$$

$$= \int_{-\pi}^{\pi} \int_{0}^{\pi/3} 9\sin\varphi \, d\varphi\, d\theta$$

$$= \int_{-\pi}^{\pi} 9\cdot(-\cos\varphi)\Big|_0^{\pi/3}$$

$$\overset{\dots}{=} \boxed{9\pi} \quad \square$$