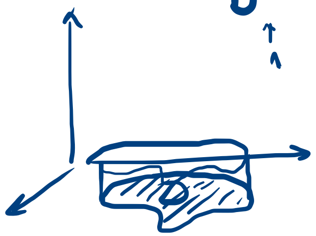
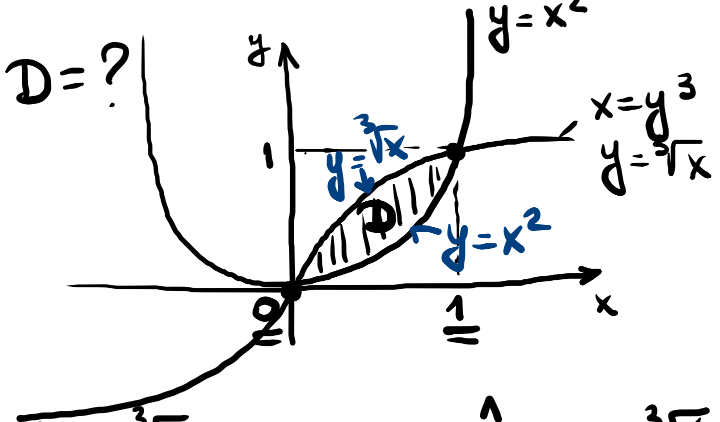
~ Двоструки интеграли - наставак ~

① $P(D) = ?$ D - између кривих $y=x^2$ и $x=y^3$

$P(D) = \iint\limits_D dx\,dy$



D = ?



$$\left.\begin{array}{l} y=x^2 \\ x=y^3 \end{array}\right\} \quad x=x^6$$

$$x=0 \lor x^5=1$$

$$\begin{array}{l} x=0 \\ \downarrow \\ y=0 \end{array} \lor \begin{array}{l} x=1 \\ \downarrow \\ y=1 \end{array}$$

$$P(D) = \iint\limits_D dx\,dy \overset{\text{фубини}}{=\!=} \int_0^1 \left(\int_{x^2}^{\sqrt[3]{x}} dy \right) dx = \int_0^1 \left(y \Big|_{x^2}^{\sqrt[3]{x}} \right) dx = \int_0^1 (\sqrt[3]{x} - x^2)\,dx$$

$$= \left(\frac{3}{4} x^{4/3} - \frac{x^3}{3} \right) \Big|_0^1 = \frac{3}{4} - \frac{1}{3} - 0 = \frac{5}{12}.$$

$$\boxed{P(D) = \frac{5}{12}}$$

□

② замени поредак интеграције: $I=\int_0^1\left(\int_{-\sqrt{2x-x^2}}^{1} f(x,y)\,dy\right)dx$

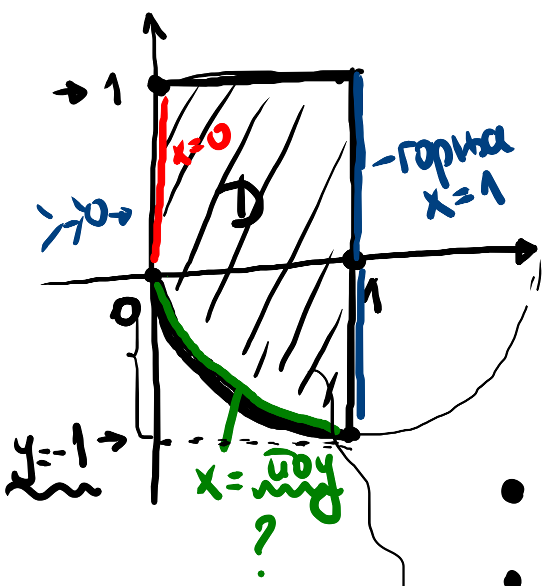
$$I=\int_{?}^{?}\left(\int_{?}^{?} f(x,y)\,dx\right)dy \leftarrow \iint_{D} f(x,y)\,dx\,dy$$

$\boxed{D=?}$

$x\in[0,1] \rightsquigarrow y\in[-\sqrt{2x-x^2},\,1]$

$$y=-\sqrt{2x-x^2}$$
$$\Leftrightarrow\ y<0 \ \text{ и } \ y^2=2x-x^2$$
$$\Leftrightarrow\ y<0 \ \text{ и } \ y^2+\underbrace{x^2-2x+1}_{(x-1)^2}=1$$
$$\Leftrightarrow\ y<0 \ \text{ и } \ (x-1)^2+y^2=1$$

круг, центар $(1,0)$
полупр. $=1$



$y\in[-1,1]$ • горња фја: $\boxed{x=1}$

доња фја:

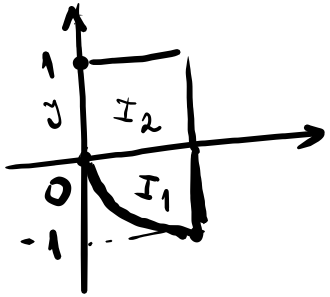
- $y\in[0,1] \Rightarrow \boxed{x=0}$
- $y\in[-1,0] \rightsquigarrow (x-1)^2+y^2=1$

$$\Leftrightarrow |x-1|=\sqrt{1-y^2}$$

☺ $\sqrt{t^2}=|t|$

$\Rightarrow x<1:\ |x-1|=1-x=\sqrt{1-y^2} \Rightarrow \boxed{x=1-\sqrt{1-y^2}}$

Γ $\iint f\,dy\,dx$ / $\int\left(\int f\,dx\right)dy$



за $y\in[-1,0]$: $x\in[1-\sqrt{1-y^2},1]$

за $y\in[0,1]$: $x\in[0,1]$

$$\Rightarrow I=\int_{-1}^{0}\left(\int_{1-\sqrt{1-y^2}}^{1} f(x,y)\,dx\right)dy+\int_{0}^{1}\left(\int_{0}^{1} f(x,y)\,dx\right)dy \qquad \blacksquare$$

② Заменити поредак интеграције: $I=\int_{0}^{2a}\left(\int_{\sqrt{2ax-x^2}}^{\sqrt{2ax}} f(x,y)\,dy\right)dx$ ($a>0$ параметар)

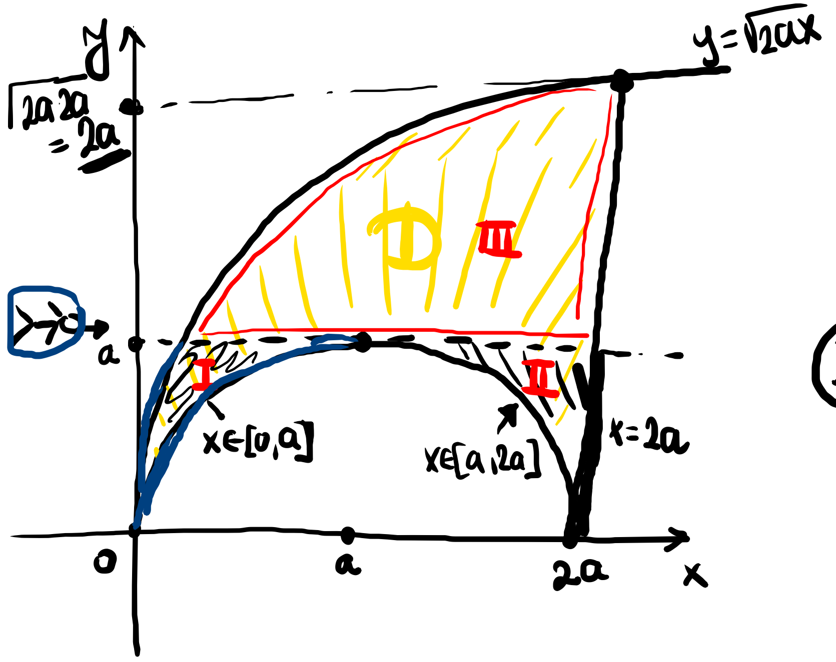
$I \rightsquigarrow \iint_D f(x,y)\,dx\,dy \qquad D=?$

- $y=\sqrt{2ax}=\sqrt{2a}\cdot\sqrt{x}$
- $y=\sqrt{2ax-x^2} \iff y>0$ и $y^2=2ax-x^2$

$\iff y>0$ и $y^2+\underbrace{x^2-2ax+a^2}_{(x-a)^2}=a^2$

$\iff y>0$ и $(x-a)^2+y^2=a^2$

круг центар: $(a,0)$
полупр.: a



$$I = \int_0^{2a}\left(\int_{\sqrt{2ax-x^2}}^{\sqrt{2ax}} f(x,y)\,dy\right)dx$$

круг: $(x-a)^2+y^2=a^2$

I) $y \in$ … $\to x \in$ …

$y\in[0,a]$ доња: $y=\sqrt{2ax} \to x=\frac{y^2}{2a}$

горња: $(x-a)^2+y^2=a^2$

$\underbrace{|x-a|}_{a-x}=\sqrt{a^2-y^2}$

овде $x\in[0,a]$

$\to a-x=\sqrt{a^2-y^2}$

$x=a-\sqrt{a^2-y^2}$

$$\frac{y^2}{2a}\le x\le a-\sqrt{a^2-y^2}$$

II) $y\in[0,a]$

доња: $(x-a)^2+y^2=a^2 \to \underbrace{|x-a|}_{x-a}=\sqrt{a^2-y^2}$

$x-a$ јер $x\geq a$

$\Rightarrow x=a+\sqrt{a^2-y^2}$

горња: $x=2a$

$$\Rightarrow a+\sqrt{a^2-y^2}\le x\le 2a$$

III) $y\in[a,2a]$ доња: $y=\sqrt{2ax}$ $x=\frac{y^2}{2a}$

горња: $x=2a$

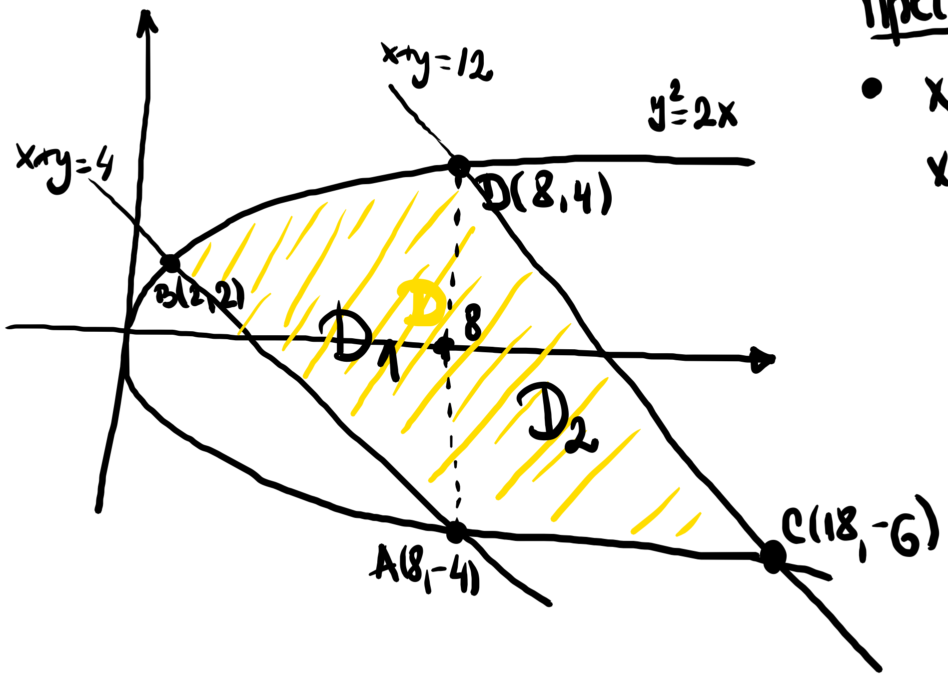
$$\frac{y^2}{2a}\le x\le 2a$$

$$\Rightarrow \quad I = \int_0^a \left(\int_{\frac{y^2}{2a}}^{a-\sqrt{a^2-y^2}} f(x,y)\,dx \right) dy + \int_0^a \left(\int_{a+\sqrt{a^2-y^2}}^{2a} f(x,y)\,dx \right) dy + \int_a^{2a} \left(\int_{\frac{y^2}{2a}}^{2a} f(x,y)\,dx \right) dy \qquad \square$$

4. $I = \iint\limits_D (x+y)\,dx\,dy$ $\quad D:$ област $y^2=2x$, $x+y=4$, $x+y=12$

$\qquad y=4-x \qquad y=12-x$

• $D = ?$



Пресечне тачке

• $x+y=4,\ y^2=2x$

$x=4-y \rightarrow y^2=8-2y$

$y^2+2y-8=0$

$(y+4)(y-2)=0$

$y=-4,\quad y=2$

$\downarrow \qquad \downarrow$

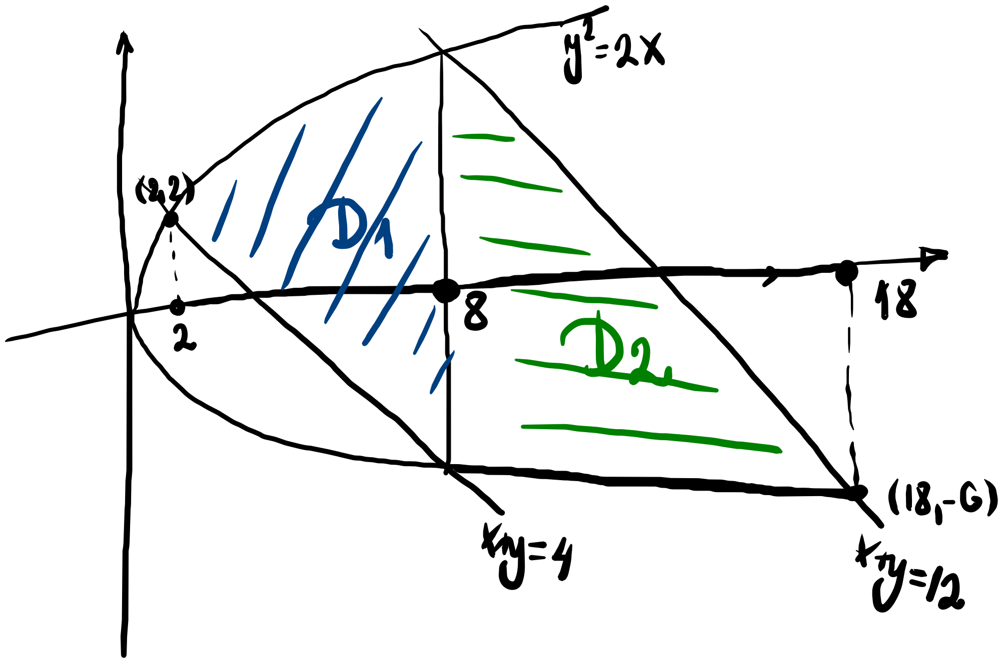
$x=8 \qquad x=2$

$A(8,-4) \quad B(2,2)$

• $x+y=12,\ y^2=2x$

$\rightsquigarrow \ \ldots$

$C(18,-6),\ D(8,4)$



D_1: $x\in[2,8]$

долна: $y=4-x$ горна: $y^2=2x$, $y>0$

$y=\sqrt{2x}$

D_2: $x\in[8,18]$

долна: $\left.\begin{matrix} y^2=2x \\ y<0 \end{matrix}\right\}$ $y=-\sqrt{2x}$ горна: $y=12-x$

$$I=\iint_{D_1}(x+y)\,dx\,dy+\iint_{D_2}(x+y)\,dx\,dy$$

$$=\int_2^8\left(\int_{4-x}^{\sqrt{2x}}(x+y)\,dy\right)dx+\int_8^{18}\left(\int_{-\sqrt{2x}}^{12-x}(x+y)\,dy\right)dx$$

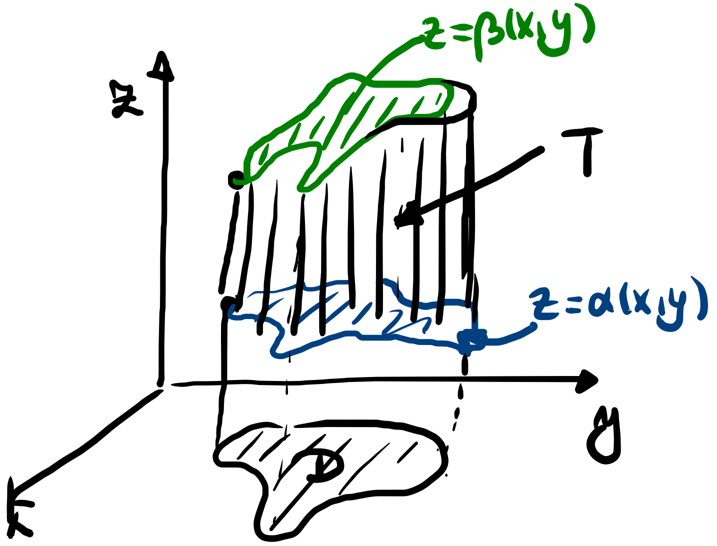
$$=\int_2^8\left(xy+\frac{y^2}{2}\right)\Bigg|_{y=4-x}^{y=\sqrt{2x}}dx+\int_8^{18}\left(xy+\frac{y^2}{2}\right)\Bigg|_{-\sqrt{2x}}^{12-x}dx=$$

$$=\int_2^8\left(x\cdot\sqrt{2x}+x-x(4-x)-\frac{(4-x)^2}{2}\right)dx+\int_8^{18}\left(x(12-x)+\frac{(12-x)^2}{2}+x\sqrt{2x}-x\right)dx=\text{рачун}\ldots$$

Тројструки интеграл

$$\iiint_T f(x,y,z)\,dx\,dy\,dz$$

$T\subset\mathbb{R}^3$



Фубинијева теорема

$T\subset\mathbb{R}^3$ мерљив и $T=\{(x,y,z)\in\mathbb{R}^3 \mid (x,y)\in D,\ \alpha(x,y)\le z\le\beta(x,y)\}$

пројекција T на $\mathbb{R}^2$

f - интеграбилна

$$\iiint_T f(x,y,z)\,dx\,dy\,dz = \iint_D \left(\int_{\alpha(x,y)}^{\beta(x,y)} f(x,y,z)\,dz\right)dx\,dy$$

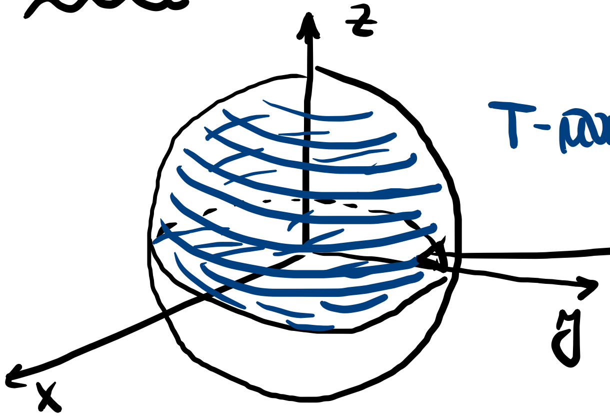
$\pm$ ако D облика:

$$D=\{(x,y)\in\mathbb{R}^2 \mid a\le x\le b,\ \varphi(x)\le y\le\psi(x)\}$$

$$\rightsquigarrow I=\int_a^b\left(\int_{\varphi(x)}^{\psi(x)}\left(\int_{\alpha(x,y)}^{\beta(x,y)} f(x,y,z)\,dz\right)dy\right)dx$$

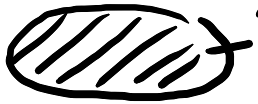
① $I = \iiint\limits_T z\,dx\,dy\,dz \qquad T = \{(x,y,z) \in \mathbb{R}^3 \mid \underbrace{x^2+y^2+z^2 \le 1}_{\text{лопта } (0,0,0)\ r=1},\ z \ge 0\}$

$T = ?$

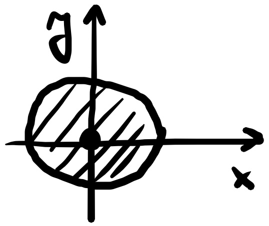


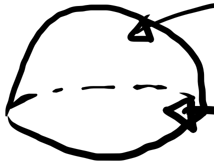
T - полулопта

у xy-равни



D - круг полупр. 1





горе: $z = \sqrt{1-x^2-y^2} \leftarrow x^2+y^2+z^2 = 1,\ z \ge 0$

доле: $z = 0$

$$\Rightarrow T = \{(x,y,z) \in \mathbb{R}^3 \mid (x,y) \in D,\ 0 \le z \le \sqrt{1-x^2-y^2}\}$$

Фубини: $$I = \iint\limits_D \left(\int_0^{\sqrt{1-x^2-y^2}} z\,dz\right) dx\,dy = \iint\limits_D \left(\frac{z^2}{2}\Big|_0^{\sqrt{1-x^2-y^2}}\right) dx\,dy$$

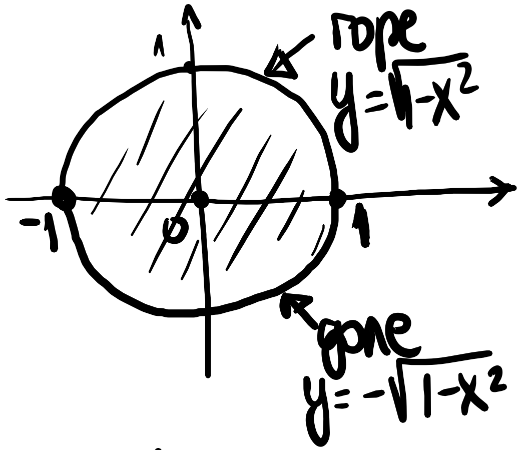
$$\Rightarrow I = \iint\limits_D \frac{1}{2}(1-x^2-y^2)\,dx\,dy$$

$$I = \frac{1}{2}\iint_D (1-x^2-y^2)\,dx\,dy$$

поларне координате

како би било без њих?

$D =$



фубини: $x\in[-1,1]$ $\quad x^2+y^2=1 \quad y^2=1-x^2 \quad |y|=\sqrt{1-x^2}$

$$\Rightarrow I = \frac{1}{2}\int_{-1}^{1}\left(\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}(1-x^2-y^2)\,dy\right)dx$$

$$\Rightarrow I = \frac{1}{2}\int_{-1}^{1}\left((1-x^2)\cdot y-\frac{y^3}{3}\right)\Bigg|_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}dx = \cancel{\frac{1}{2}}\int_{-1}^{1}\left((1-x^2)\cdot\cancel{2}\sqrt{1-x^2}-\cancel{2}\cdot\frac{1}{3}\left(\sqrt{1-x^2}\right)^3\right)dx$$

$$=\int_{-1}^{1}\frac{2}{3}(1-x^2)^{3/2}dx \overset{\text{парна}}{=} 2\cdot\int_{0}^{1}\frac{2}{3}(1-x^2)^{3/2}dx \;\frac{x=\sin t}{t\in[0,\pi/2]}\; \frac{4}{3}\int_0^{\pi/2}(\underbrace{1-\sin^2 t}_{\cos^2 t})^{3/2}\cdot\cos t\,dt$$

$dx=\cos t\,dt$

$(\cos^2 t)^{3/2}=|\cos t|^3=(\cos t)^3$

$$=\frac{4}{3}\int_0^{\pi/2}\cos^4 t\,dt = \frac{4}{3}\int_0^{\pi/2}\left(\frac{1+\cos 2t}{2}\right)^2 dt = \frac{1}{3}\int_0^{\pi/2}\left(1+2\cos 2t+(\cos 2t)^2\right)dt$$

$$=\frac{1}{3}\int_0^{\pi/2}\left(1+2\cos 2t+\frac{1+\cos 4t}{2}\right)dt = \frac{1}{3}\left(\frac{3}{2}t+\sin 2t+\frac{1}{8}\sin 4t\right)\Bigg|_0^{\pi/2} = \frac{\pi}{4} \quad \square$$

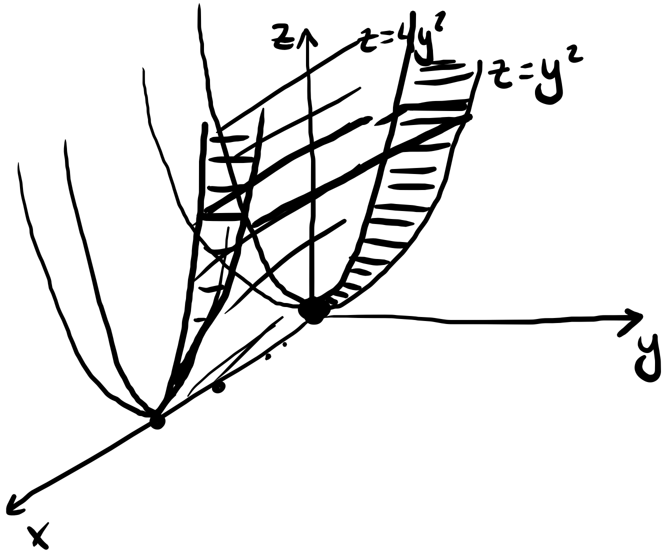
② $I = \iiint\limits_T x^2\,dx\,dy\,dz$ T: огр. са: $\underbrace{z = y^2,\ z = 4y^2}$ (1)

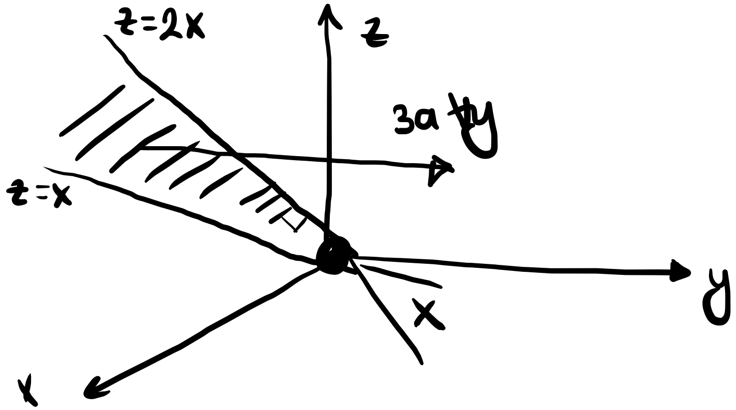
$y \ge 0$

$\underbrace{z = x,\ z = 2x}$, $z = h$ $(h > 0)$

(1): $\underbrace{y^2 \le z \le 4y^2} \Rightarrow \underline{z \ge 0}$ $\Rightarrow$ $\underbrace{x \le z \le 2x}$

$\underbrace{\frac{z}{2} \le x \le z}$





- $\boxed{z \in [0, h]}$
- $y \ge 0$, $y^2 \le z \le 4y^2$ $\rightsquigarrow$ $\boxed{\frac{\sqrt{z}}{2} \le y \le \sqrt{z}}$
- $\boxed{\frac{z}{2} \le x \le z}$

$\Rightarrow I = \int_0^h \left(\int_{\frac{\sqrt{z}}{2}}^{\sqrt{z}} \left(\int_{\frac{z}{2}}^{z} x^2\,dx \right) dy \right) dz$

$=$ рачун...