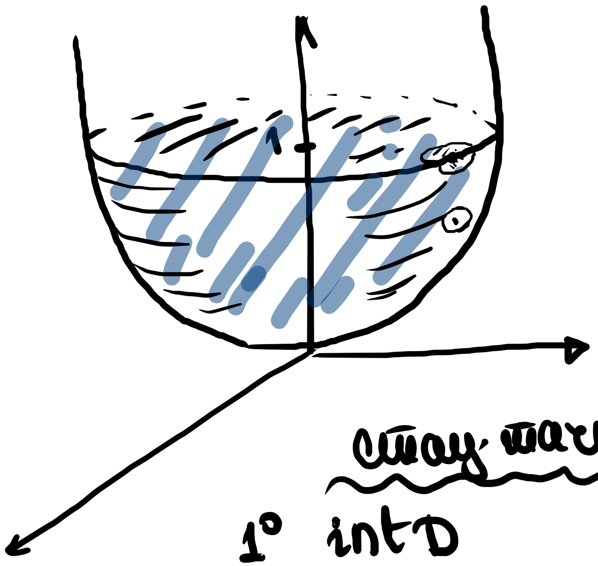
Условни екстремуми - наставак

1. $f(x,y,z)=x+y+z$ $\quad D=\{(x,y,z)\in\mathbb{R}^3 \mid x^2+y^2\le z\le 1\}$ Наћи min и max f на D.



$z=x^2+y^2$ ПАРАБОЛОИД $\quad z\le 1$

$\geq$ унутр. + гр.

D - затв. огр. $\Rightarrow$ D компактан, f непр. $\Big\}$ $\xRightarrow{\text{Вајерштрас}}$ f достиже min, max на D

стац. тачке

1° $\mathrm{int}D$

∂D $\begin{cases} 2° \text{ део параболоида} \\ 3° \text{ кружница} \\ 4° \text{ унутрашњост круга} \end{cases}$

$\boxed{1° \ \mathrm{int}D}$ $\quad \nabla f \overset{?}{=} \vec{0}$

$f'_x=1,\ f'_y=1,\ f'_z=1$

$\nabla f=(1,1,1)\neq\vec{0}$

$\Rightarrow$ нема стац. т. у $\mathrm{int}D$.

$$\begin{cases} f(x,y,z)=x+y+z \\ D:\ x^2+y^2\le z\le 1 \end{cases}$$

2°

$z=x^2+y^2,\ z<1$

$\varphi(x,y,z)=x^2+y^2-z=0$

Лагранжова функција: $F(x,y,z,\lambda)=f(x,y,z)+\lambda\cdot\varphi(x,y,z)=x+y+z+\lambda\cdot(x^2+y^2-z)$

$F'_x=1+2\lambda x=0$

$F'_y=1+2\lambda y=0$

$F'_z=1-\lambda=0 \Rightarrow \boxed{\lambda=1}$

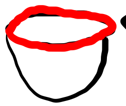
$F'_\lambda=x^2+y^2-z=0$

$\Rightarrow \boxed{\begin{matrix} x=-\frac{1}{2} \\ y=-\frac{1}{2}\end{matrix}} \xrightarrow{z=x^2+y^2} \boxed{z=\frac{1}{2}}$

стац. тачка $\boxed{M_1\left(-\frac{1}{2},-\frac{1}{2},\frac{1}{2}\right)}$

$f(M_1)=-\frac{1}{2}$

3°



$\leftarrow z=1$ $\quad \underset{\text{услов}}{x^2+y^2=1}$ $\xrightarrow{z=1}$ ми посматрамо:

$f_1(x,y)=x+y+1$

Лагр. ф-ја: $F(x,y,\lambda)=x+y+1+\lambda\cdot(x^2+y^2-1)$

$$\left.\begin{matrix} F'_x=1+2\lambda x=0 \\ F'_y=1+2\lambda y=0 \end{matrix}\right\}\ x=y=-\frac{1}{2\lambda}$$

$F'_\lambda=x^2+y^2-1=0$

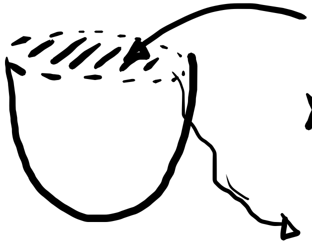
$2\cdot\frac{1}{4\lambda^2}=1$

$2\lambda^2=1$

$\lambda_1=\frac{1}{\sqrt{2}} \rightarrow M_2\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}},1\right)$ $\quad f(M_2)=\sqrt{2}+1$

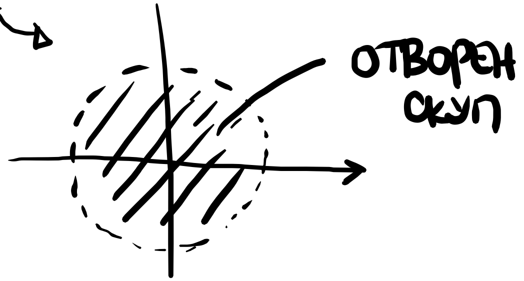
$\lambda_2=-\frac{1}{\sqrt{2}} \rightarrow M_3\left(-\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}},1\right)$ $\quad f(M_3)=-\sqrt{2}+1$

4°



$z = 1 \to f(x,y,1) = x+y+1 =: g(x,y)$

$x^2+y^2<1$



$\nabla g \overset{?}{=} \vec{0}$

$\left.\begin{array}{l} g'_x = 0 \\ g'_y = 0 \end{array}\right\}$?

$g'_x = 1$

$g'_y = 1$

$\nabla g = (1,1)$

$\Rightarrow$ нема стац. тачака!

$\underline{\underline{D}}$ кандидати: $f(M_1), f(M_2), f(M_3)$

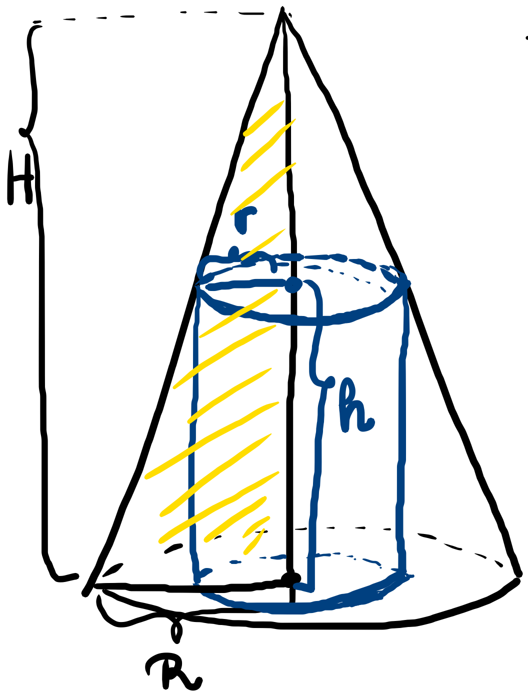
$-\frac{1}{2}$ (min), $\sqrt{2}+1$ (max), $-\sqrt{2}+1$

$$\boxed{\max_D f = \sqrt{2}+1}$$

$$\boxed{\min_D f = -\frac{1}{2}}$$

$\square$

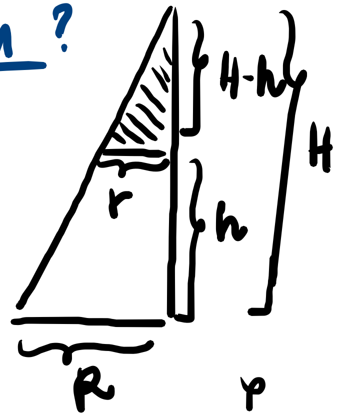
2) Дата је купа: висина H, R-полупр. основе
уписати ваљак max запремине тд. основа ваљка лежи у основи купе.



h-висина ваљка, r-полупр. основе $\quad 0<r<R,\ 0<h<H$

$\underline{V_{max}=?}\quad V=h\cdot r^2\pi \qquad f(\overset{x}{r},\overset{y}{h})=r^2h \quad \underline{f_{max}=?} \qquad f(x,y)=x^2\cdot y\ \ \underline{max}$

услови?



$$\boxed{\frac{r}{R}=\frac{H-h}{H}} \Leftrightarrow H\cdot r = H\cdot R - h\cdot R \Leftrightarrow H\cdot R - \underset{y}{h}\cdot R - H\cdot \underset{x}{r} = 0$$

$$\boxed{\underbrace{HR-R\cdot y-H\cdot x}_{\varphi(x,y)}=0}$$

Лагранжова ф-ја: $F(x,y,\lambda)=\overbrace{x^2\cdot y}^{f}+\lambda\cdot(\overbrace{HR-Ry-H\cdot x}^{\varphi})$

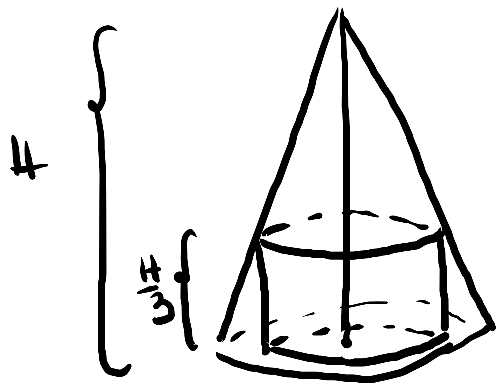
$$\left.\begin{aligned} F'_x &= 2xy-\lambda H=0 \quad /\cdot R\\ F'_y &= x^2-\lambda R=0 \quad /\cdot H\end{aligned}\right\} - \Rightarrow 2Rxy-H\cdot x^2=0 \qquad x\cdot(2Ry-H\cdot x)=0 \overset{\text{за max}}{\Rightarrow} 2Ry-Hx=0$$

сигурно V није max за $x=0$

$$F'_\lambda = HR-Ry-Hx=0$$

$$\boxed{y=\frac{H}{3},\ x=\frac{2R}{3}}$$

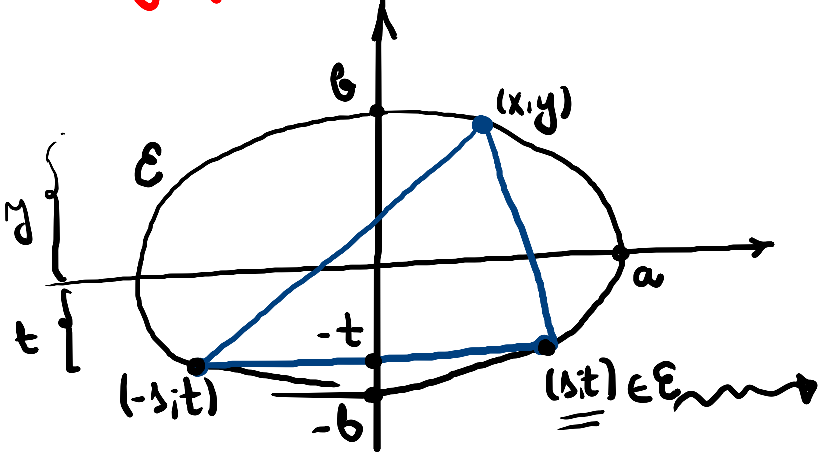
V_{max} za $x = r = \frac{2R}{3}$, $h = y = \frac{H}{3}$



$V_{max} = h \cdot r^2 \pi$

$$\underline{\underline{V_{max}}} = \frac{4 \cdot R^2 \cdot H \pi}{27}$$

③ У елипсу уписати △ макс. површине тд је једна страна △ $\parallel$ x-оси елипсе.



$\varepsilon: \frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

тражимо $T: (s,-t), (-s,-t), (x,y)$

$0<s<a, \quad 0<t<b \qquad \frac{x^2}{a^2}+\frac{y^2}{b^2}=1, \ y>0$

$\frac{s^2}{a^2}+\frac{t^2}{b^2}=1 \overset{s>0}{\Rightarrow} \frac{s^2}{a^2}=\frac{b^2-t^2}{b^2} \overset{s>0}{\Rightarrow} \boxed{s=\frac{a}{b}\cdot\sqrt{b^2-t^2}}$

$\boxed{P(T)} = \frac{2s\cdot(y+t)}{2} = s\cdot(y+t) = \frac{a}{b}\cdot\sqrt{b^2-t^2}\cdot(y+t) \qquad \max = ?$

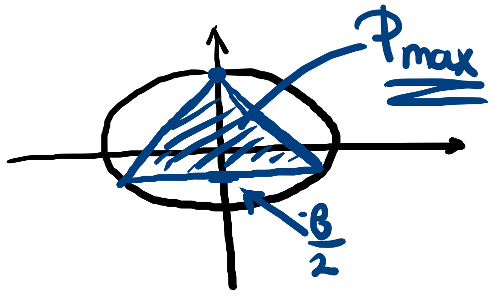
Довољно је наћи max: $f(x,y,t) = (b^2-t^2)\cdot(y+t)^2$

услов: $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \iff \underbrace{b^2x^2+a^2y^2-a^2b^2}_{\varphi(x,y)}=0 \qquad \varphi(x,y)=0$

! могли смо одмах видети да $x=0$ за P_{max} !

Лагранж: $F(x,y,t,\lambda) = (b^2-t^2)(y+t)^2+\lambda\cdot(b^2x^2+a^2y^2-a^2b^2)$

за вежбу: $F'_x=0 \quad F'_t=0 \quad F'_y=0 \quad F'_\lambda=0 \quad \ldots \rightsquigarrow \boxed{x=0, y=b, t=\frac{b}{2}}$



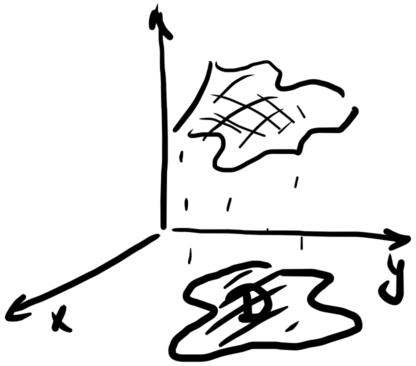
~Вишеструки интеграли~

Двоструки интеграли

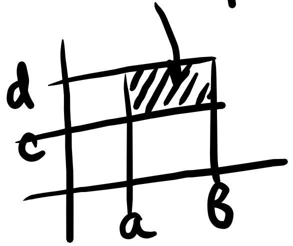
$$\iint f(x,y)\,dxdy$$

D- мерљив

$D \subset \mathbb{R}^2$



⊛ $D=\Pi$-правоугаоник



$\boxed{\text{Фубини}}$ f-непр. на П

$$\iint\limits_{\Pi} f(x,y)\,dxdy = \int_a^b\left(\int_c^d f(x,y)\,dy\right)dx$$

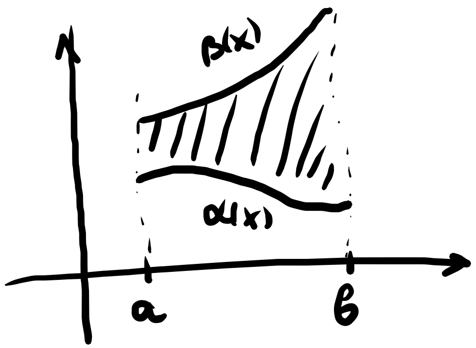
$$= \int_c^d \Big(\underbrace{\int_a^b f(x,\underset{=}{y})\,dx}_{\varphi(y)}\Big)\,dy$$

"const"

⊛ $D=\{(x,y)\in\mathbb{R}^2 \mid a\le x\le b,\ \alpha(x)\le y\le\beta(x)\}$

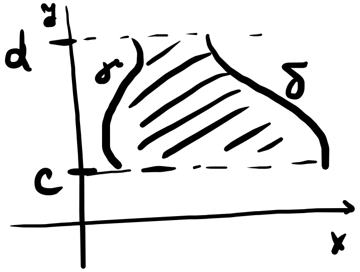
f-интеграбилна D

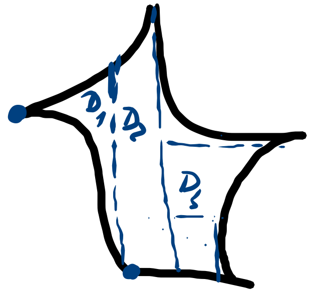
$$\iint\limits_{D} f(x,y)\,dxdy = \int_a^b\left(\int_{\alpha(x)}^{\beta(x)} f(x,y)\,dy\right)dx$$



⊛ $D=\{(x,y)\in\mathbb{R}^2 \mid c\le y\le d,\ \gamma(y)\le x\le \delta(y)\}$

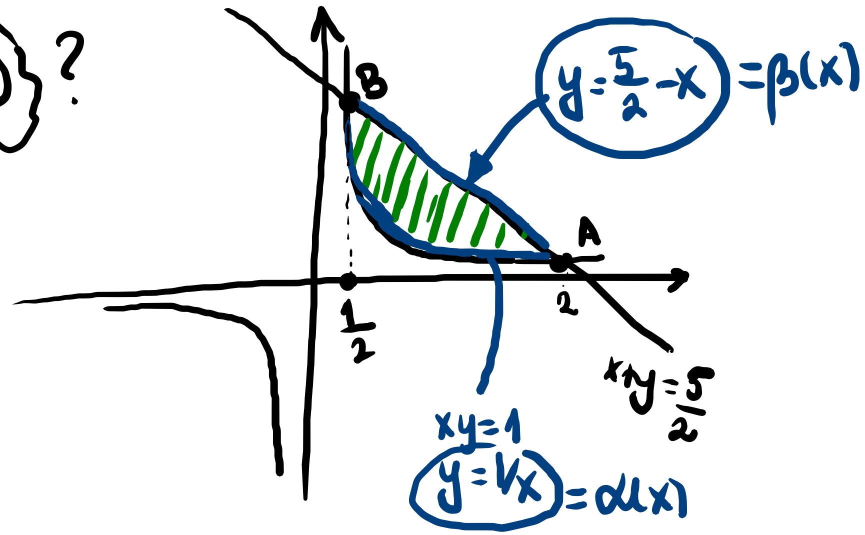
$$\iint_D f(x,y)\,dx\,dy=\int_c^d\left(\int_{\gamma(y)}^{\delta(y)} f(x,y)\,dx\right)dy$$





1. $I=\iint_D xy\,dx\,dy$ $\quad D:\ x\cdot y=1,\ x+y=\frac{5}{2}$

D ?



Граница за x ?

пресеци α и β: $\left.\begin{array}{l} xy=1 \\ x+y=\frac{5}{2} \end{array}\right\}$

$\frac{5}{2}-x=\frac{1}{x} \quad /\cdot x$

$x^2-\frac{5}{2}x+1=0 \qquad 2x^2-5x+2=0$

$x_1=2 \rightarrow y_1=\frac{1}{2}$

$x_2=\frac{1}{2} \rightarrow y_2=2$

⇒ пресечне т.: $A(2,\frac{1}{2}),\ B(\frac{1}{2},2)$

Фубинијева т. ⇝ $D=\{(x,y)\in\mathbb{R}^2 \mid \frac{1}{2}\le x\le 2,\ \frac{1}{x}\le y\le \frac{5}{2}-x\}$

⇝ $I=\int_{1/2}^{2}\left(\int_{1/x}^{\frac{5}{2}-x} x\cdot y\,dy\right)dx$

$$I = \int_{1/2}^{2} \left(\int_{1/x}^{\frac{5}{2}-x} x \cdot y \, dy \right) dx$$

$$= \int_{1/2}^{2} x \cdot \frac{y^2}{2} \Big|_{1/x}^{\frac{5}{2}-x} dx = \int_{1/2}^{2} x \cdot \left(\frac{1}{2}\left(\frac{5}{2}-x\right)^2 - \frac{1}{2} \cdot \frac{1}{x^2} \right) dx$$

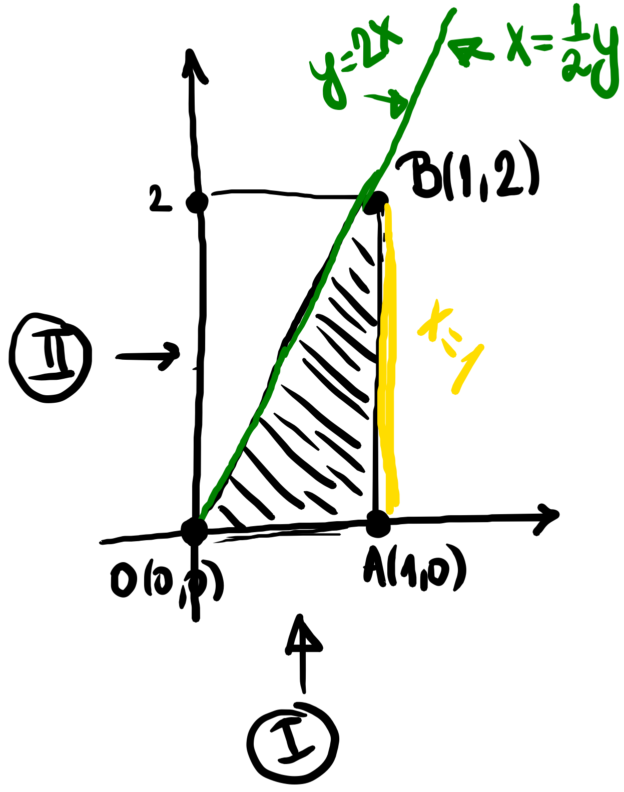
$$= \frac{1}{2} \int_{1/2}^{2} \left(\frac{25}{4}x - 5x^2 + x^3 - \frac{1}{x} \right) dx$$

$$= \frac{1}{2} \left(\frac{25}{4} \cdot \frac{x^2}{2} - \frac{5}{3}x^3 + \frac{x^4}{4} - \ln x \right) \Big|_{1/2}^{2}$$

= parynt ☺ ... $\square$

② D – троугао: $O(0,0)$, $A(1,0)$, $B(1,2)$

Одредити границе за оба поретка у $I=\iint\limits_D f(x,y)\,dx\,dy$



• I **прво по y**

$y = ?(x)$

горња ф-ја $y=2x$

$$I=\int_0^1\left(\int_0^{2x} f(x,y)\,dy\right)dx$$

доња ф-ја $y=0$

II **прво по x**

$x = ?(y)$

$x=1$

$$I=\int_0^2\left(\int_{\frac{1}{2}y}^{1} f(x,y)\,dx\right)dy$$

$x=\frac{1}{2}y$