~ Пресликавања са вредностима у $\mathbb{R}^n$ ~

$F:\mathbb{R}^m\to\mathbb{R}^n \qquad F=(f_1,f_2,\dots,f_n) \qquad f_i:\mathbb{R}^m\to\mathbb{R}$

F диференцијабилна у $x_0\in\mathbb{R}^m$ ако $\exists$ лин. пресл. $L:\mathbb{R}^m\to\mathbb{R}^n$ тд.

$$F(x_0+h)=F(x_0)+\underbrace{L(h)}_{=\,A_L\cdot h}+o(h), \quad h\to 0 \iff (h_1,\dots,h_m)\to(\underbrace{0,\dots,0}_{m})$$

$$n\underbrace{\begin{bmatrix} \cdots \\ \cdots \end{bmatrix}}_{m}\cdot\begin{bmatrix} h_1 \\ \vdots \\ h_m\end{bmatrix}_{m\times 1} \qquad [A_L]_{n\times m}$$

L се назива **ИЗВОД** ф-је F у x_0 $\qquad L=dF(x_0),\ F'(x_0),\ \dfrac{dF}{dx}(x_0)$

A_L – Јакобијева матрица F у x_0

$$A_L=\begin{bmatrix} \frac{\partial f_1}{\partial x_1}(x_0) & \frac{\partial f_1}{\partial x_2}(x_0) & \cdots & \frac{\partial f_1}{\partial x_m}(x_0) \\ \vdots & & & \vdots \\ \frac{\partial f_n}{\partial x_1}(x_0) & \frac{\partial f_n}{\partial x_2}(x_0) & \cdots & \frac{\partial f_n}{\partial x_m}(x_0)\end{bmatrix}_{n\times m}$$

$m=n$

$J_F=\det A$

Јакобијан

Извод сложене ф-је

$A \subset \mathbb{R}^m \quad f: A \to \mathbb{R}^n \qquad \mathbb{R}^m \xrightarrow{f} \mathbb{R}^n \xrightarrow{g} \mathbb{R}^k$

$B \subset \mathbb{R}^n \quad g: B \to \mathbb{R}^k$

$x_0 \in A$, $y_0 \in B$ $\quad f(x_0) = y_0 \qquad$ f диф. у x_0, g диф. у y_0

$\Rightarrow$ $g \circ f$ диф у x_0 и

$$d(g\circ f)(x_0) = dg(y_0) \circ df(x_0)$$

Матрице: $\quad A_{g\circ f(x_0)} = A_{g(f(x_0))} \cdot A_{f(x_0)}$

1. $f:(r,\varphi,z)\to(\underbrace{r\cos\varphi}_{f_1},\underbrace{r\sin\varphi}_{f_2},\underbrace{z}_{f_3})$; $g:(x,y,z)\to(\underbrace{\sqrt{x^2+y^2}}_{g_1},\underbrace{\operatorname{arctg}\frac{y}{x}}_{g_2},\underbrace{z}_{g_3})$

а) Јакобијева матрица g ?

б) — || — $(g\circ f)$ на скупу $D=\{(r,\varphi,z)\in\mathbb{R}^3 \mid \underline{1<r\leq 5},\ \varphi\in[0,\frac{9\pi}{5}),\ |z|\leq 2\}$

(а)
$$A_g=\begin{bmatrix}\frac{\partial g_1}{\partial x} & \frac{\partial g_1}{\partial y} & \frac{\partial g_1}{\partial z}\\ \frac{\partial g_2}{\partial x} & \frac{\partial g_2}{\partial y} & \frac{\partial g_2}{\partial z}\\ \frac{\partial g_3}{\partial x} & \frac{\partial g_3}{\partial y} & \frac{\partial g_3}{\partial z}\end{bmatrix}=\begin{bmatrix}\frac{1}{2}\frac{2x}{\sqrt{x^2+y^2}} & \frac{1}{2}\frac{2y}{\sqrt{x^2+y^2}} & 0\\ \frac{1}{1+\frac{y^2}{x^2}}\cdot\frac{-y}{x^2} & \frac{1}{1+\frac{y^2}{x^2}}\cdot\frac{1}{x} & 0\\ 0 & 0 & 1\end{bmatrix}=\begin{bmatrix}\frac{x}{\sqrt{x^2+y^2}} & \frac{y}{\sqrt{x^2+y^2}} & 0\\ \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2} & 0\\ 0 & 0 & 1\end{bmatrix}$$

(б) $(g\circ f)'=g'\circ \underline{\underline{f'}}$

$$A_f=\begin{bmatrix}\frac{\partial f_1}{\partial r} & \frac{\partial f_1}{\partial \varphi} & \frac{\partial f_1}{\partial z}\\ — & — & —\\ — & — & —\end{bmatrix}=\begin{bmatrix}\cos\varphi & -r\sin\varphi & 0\\ \sin\varphi & r\cos\varphi & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$A_g = \begin{bmatrix} \frac{x}{\sqrt{x^2+y^2}} & \frac{y}{\sqrt{x^2+y^2}} & 0 \\ -\frac{y}{x^2+y^2} & \frac{x}{x^2+y^2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \underline{A_f} = \begin{bmatrix} \cos\varphi & -r\sin\varphi & 0 \\ \sin\varphi & r\cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$(g\circ f)' = g'\circ f'$ → матрице $\boxed{A_{g\circ f} = A_g \cdot A_f}$

у тачки $f(r,\varphi,z) = (\underbrace{r\cos\varphi}_{x}, \underbrace{r\sin\varphi}_{y}, \underbrace{z}_{z})$ у (r,φ,z)

$$A_g(\underbrace{r\cos\varphi}_{x}, \underbrace{r\sin\varphi}_{y}, z) = [\,?\,]_{3\times 3}$$

$x^2+y^2 = r^2$

$\sqrt{x^2+y^2} = \sqrt{r^2} = |r| = r$ на D $r>0$

$$\underline{A_g(r\cos\varphi, r\sin\varphi, z)} = \begin{bmatrix} \cos\varphi & \sin\varphi & 0 \\ -\frac{1}{r}\sin\varphi & \frac{\cos\varphi}{r} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A_{g\circ f} = A_g(r\cos\varphi, r\sin\varphi, z)\cdot A_f(r,\varphi,z)$$

$$= \begin{bmatrix} \cos\varphi & \sin\varphi & 0 \\ -\frac{\sin\varphi}{r} & \frac{\cos\varphi}{r} & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \cos\varphi & -r\sin\varphi & 0 \\ \sin\varphi & r\cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Парц. изводи вишег реда

(A1) $f'(x), f''(x), \dots$

(A3) $f:\mathbb{R}^2\to\mathbb{R}$ I реда: f'_x, f'_y II реда: $(f'_x)'_x,\ (f'_x)'_y,\ (f'_y)'_x,\ (f'_y)'_y$

$$\begin{array}{cccc} (f'_x)'_x & (f'_x)'_y & (f'_y)'_x & (f'_y)'_y \\ \| & \| & \| & \| \\ f''_{xx} & \underline{\underline{f''_{xy}}} & \underline{\underline{f''_{yx}}} & f''_{yy} \end{array}$$

☺ f''_{xy} и f''_{yx} постоје у околини (x_0,y_0) и непр. у (x_0,y_0)
$\Rightarrow f''_{xy}(x_0,y_0)=f''_{yx}(x_0,y_0)$

Правило ланца

$$\underset{\mathbb{R}^3}{(x,y,z)} \xrightarrow{g} \underset{\mathbb{R}^3}{(\underset{g_1}{u},\underset{g_2}{v},\underset{g_3}{w})} \xrightarrow{f} \left(f_1(u,v,w),\ f_2(u,v,w)\right)\in\mathbb{R}^2$$

$(f_1)'_x = ?$ $(f_2)'_x = ?$

$\boxed{f\circ g}$ $\mathbb{R}^3\to\mathbb{R}^2$

$$\begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v} & \frac{\partial f_1}{\partial w} \\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v} & \frac{\partial f_2}{\partial w} \end{bmatrix}}_{\{f'\}} \cdot \underbrace{\begin{bmatrix} u'_x & u'_y & u'_z \\ v'_x & v'_y & v'_z \\ w'_x & w'_y & w'_z \end{bmatrix}}_{\{g'\}}$$

$$\frac{\partial f_1}{\partial y} = (f_1)'_y = \frac{\partial f_1}{\partial u}\cdot u'_y + \frac{\partial f_1}{\partial v}\cdot v'_y + \frac{\partial f_1}{\partial w}\cdot w'_y$$

② Трансформисати узимајући у обзир: $\quad 2z''_{xx} + z''_{xy} - z''_{yy} + z'_x + z'_y = 0 \quad (*)$

+ сви мешовити изводи једнаки

$u = x + 2y + 2$

$v = x - y - 1 \qquad z = z(u, v)$

$$\boxed{\begin{array}{ll} u'_x = 1 & u'_y = 2 \\ v'_x = 1 & v'_y = -1 \end{array}}$$

$$z'_x = z'_u \cdot u'_x + z'_v \cdot v'_x \qquad z'_x = [z'_u \;\; z'_v] \cdot \begin{bmatrix} u'_x \\ v'_x \end{bmatrix}$$

$$\boxed{z'_x = z'_u + z'_v} \quad (1)$$

$$z'_y = z'_u \cdot u'_y + z'_v \cdot v'_y = \boxed{z'_u \cdot 2 - z'_v} \quad (2)$$

$$z''_{xx},\; z''_{xy} = z''_{yx},\; z''_{yy} = ?$$

$$(1)'_x : \boxed{z''_{xx}} = (z'_x)'_x \overset{(1)}{=} (z'_u + z'_v)'_x = (z'_u + z'_v)'_u \cdot u'_x + (z'_u + z'_v)'_v \cdot v'_x =$$

$$= z''_{uu} \cdot \underbrace{u'_x}_{1} + \underbrace{z''_{vu} \cdot u'_x + z''_{uv} \cdot v'_x}_{} + z''_{vv} \cdot \underbrace{v'_x}_{1} = \boxed{z''_{uu} + 2z''_{uv} + z''_{vv}}$$

$$(1)'_y \to (z''_{xy}) = (z'_x)'_y = (z'_u + z'_v)'_y = z''_{uu} \cdot \underbrace{u'_y}_{2} + z''_{uv} \cdot \underbrace{v'_y}_{-1} + z''_{vu} \cdot \underbrace{u'_y}_{2} + z''_{vv} \cdot \underbrace{v'_y}_{-1} = \boxed{2z''_{uu} + z''_{uv} - z''_{vv}}$$

$$(2)'_y \to z''_{yy} = (z'_y)'_y = \dots = \boxed{4z''_{uu} - 4z''_{uv} + z''_{vv}}$$

Заменим се у (*):

$$2\cancel{z''_{uu}}+4z''_{uv}+\cancel{2z''_{vv}}+\cancel{2z''_{uu}}+z''_{uv}-\cancel{z''_{vv}}-\cancel{4z''_{uu}}+4z''_{uv}-\cancel{z''_{vv}}+z'_u+\cancel{z'_v}+2z'_u-\cancel{z'_v}=0$$

$$9z''_{uv}+3z'_u=0 \quad /:3 \quad \to \quad \boxed{3z''_{uv}+z'_u=0}$$

Тејлоров полином: $f(x,y)$ $\exists$ п. изв. до реда n + непр. сви

степена n у (x_0,y_0)

$$\underbrace{P_n(x_0,y_0)}_{\text{име}}(x,y)=f(x_0,y_0)+\frac{1}{1!}\Big(f'_x(x_0,y_0)\cdot\overbrace{(x-x_0)}^{h}+f'_y(x_0,y_0)\cdot\overbrace{(y-y_0)}^{k}\Big)$$

$$+\frac{1}{2!}\Big(f''_{xx}(x_0,y_0)\cdot(x-x_0)^2+2\cdot f''_{xy}(x_0,y_0)\cdot(x-x_0)(y-y_0)+f''_{yy}(x_0,y_0)\cdot(y-y_0)^2\Big)$$

$$+\dots$$

$$+\frac{1}{n!}\cdot\sum_{j=0}^{n}\binom{n}{j}\frac{\partial f}{\partial^{n-j}x\,\partial^{j}y}(x_0,y_0)\cdot(x-x_0)^{n-j}\cdot(y-y_0)^{j}$$

③ $f(x,y) = e^{2x}\cdot \sin 3y \quad n=3, \quad A=(0,0)$

вредности у $(0,0)$

$f(0,0)=0$

$f'_x = 2e^{2x}\sin 3y \qquad f'_x(0,0) = \underline{0}$

$f'_y = 3e^{2x}\cos 3y \rightarrow \underline{3}$

$f''_{xx} = 4e^{2x}\sin 3y \rightarrow 0,$

$f''_{xy} = 6e^{2x}\cos 3y \quad = 6,$

$f''_{yy} = (f'_y)'_y = -9\, e^{2x}\sin 3y \quad = 0,$

$f'''_{xxx} = (f''_{xx})'_x = 8\cdot e^{2x}\sin 3y \quad = 0,$

$f'''_{xxy} = 12e^{2x}\cdot\cos 3y \quad = 12,$

$f'''_{xyy} = (f''_{xy})'_y = -18e^{2x}\sin 3y \quad = 0,$

$f'''_{yyy} = -27\cdot e^{2x}\cos 3y \quad = -27,$

! су мешовити једнаки

Тејлор:

$$P_3(0,0)(x,y) = 0 + \frac{1}{1!}\left(0\cdot \overset{x}{(x-0)} + 3\cdot \overset{y}{(y-0)}\right)$$

$$+ \frac{1}{2!}\cdot\left(0\cdot x^2 + 2\cdot 6\cdot xy + 0\cdot y^2\right)$$

$$+ \frac{1}{3!}\cdot\left(0\cdot x^3 + \binom{3}{1}\cdot \underset{f'''_{xxy}}{12}\cdot x^2y + \binom{3}{1}\cdot 0\cdot xy^2 + (-27)\cdot y^3\right)$$

$$\boxed{P_3(x,y) = 3y + 6xy + 6x^2y - \frac{9}{2}y^3}$$

☺ "производ пол. за e^{2x} и пол. за $\sin 3y$

! чланови до реда 3

4) Имплицитна ф-ја:

$z=z(x,y)$ $2x^2-y^2-z^2-2xy+yz^2=0,\ z>0$ (*)

$f(x)=$ ф-ја од x експлицитно

$1+(f(x))^2=\sin x$ имплицитно

Наћи Тејлоров полином степена 2 у $A(0,2)$ ф-је $z=z(x,y)$.

$z(0,2)$ $z'_x, z'_y, z''_{xx}, z''_{xy}, z''_{yy}$ у $(0,2)$?

(*) $x=0, y=2$: $0-4-z^2-0+2z^2=0$ $z^2=4 \leftarrow (z(0,2))^2=4 \xRightarrow{z>0} \boxed{z(0,2)=2}$

$(*)'_x$: $4x-0-2z\cdot z'_x-2y+y\cdot 2z\cdot z'_x=0$: $\boxed{4x-2z\cdot z'_x+2y\cdot z\cdot z'_x-2y=0}$ (**)

$A(0,2) \to x=0, y=2, z=2$ $\Rightarrow -4z'_x-4+8z'_x=0 \Rightarrow \boxed{z'_x(0,2)=1}$

$(*)'_y$: $-2y-2z\cdot z'_y-2x+z^2+y\cdot 2z\cdot z'_y=0$ (***) $\xRightarrow[z=2]{x=0,y=2} \dots \boxed{z'_y(0,2)=0}$

$z''_{xx}=?$ у $(0,2)$ из $(**)\big|'_x$: $4-2\cdot z'_x\cdot z'_x-2\cdot z\cdot z''_{xx}+2y(z'_x)^2+2y\cdot z\cdot z''_{xx}=0$

уврстимо у $(0,2)$: $\left.\begin{matrix} z=2 \\ z'_x=1 \end{matrix}\right\}$ $4-2\cdot 1^2-2\cdot 2\cdot z''_{xx}+2\cdot 2\cdot 1^2+8z''_{xx}=0$

$6+4z''_{xx}=0$ $\boxed{z''_{xx}(0,2)=-3/2}$

$z''_{xy} \to$ из $(**)'_y$ $\rightsquigarrow$ $\underline{z''_{xy}(0,2) = -1/2}$

$z''_{yy} \to$ из $(***)'_y$ $\rightsquigarrow$ $\underline{z''_{yy}(0,2) = 1/2}$

$\begin{bmatrix} z = 2 \\ z'_x = 1 \\ z'_y = 0 \\ \underline{z''_{xx}} = -3/2 \end{bmatrix}$

тейлоров полином у $(0,2)$:

$$P_2(0,2)(x,y) = 2 + \frac{1}{1!}\cdot\left(1\cdot(x-0) + 0\cdot(y-2)\right) + \frac{1}{2!}\left(-\frac{3}{2}\cdot x^2 + \binom{2}{1}\cdot -\frac{1}{2}\cdot x\cdot(y-2) + \frac{1}{2}\cdot(y-2)^2\right)$$

$$= 2 + x - \frac{3}{4}x^2 - \frac{1}{2}x(y-2) + \frac{1}{4}(y-2)^2$$

$$\overset{\text{среди}}{=} \underline{3 + 2x - y - \frac{3}{4}x^2 - \frac{1}{2}xy + \frac{1}{4}y^2}$$

Прошли задатак:

$$f(x,y)=\begin{cases} e^{\frac{x^3}{x^2+y^2}}, & (x,y)\neq(0,0) \\ 1, & (x,y)=(0,0) \end{cases}$$