~ Парцијални изводи и диференцијабилност ~

$f: \mathbb{R}^n \to \mathbb{R} \quad (x_1, \dots, x_n) \in \mathbb{R}$

$i \in \{1, \dots, n\}: \quad f'_{x_i}(x) = \lim\limits_{h \to 0} \dfrac{f(x_1, \dots, x_{i-1}, x_i + h, x_{i+1}, \dots, x_n) - f(x)}{h} \qquad x = (x_1, \dots, x_n)$

$= \dfrac{\partial f}{\partial x_i}(x)$

$f: \mathbb{R}^2 \to \mathbb{R} \qquad (x_0, y_0) \in \mathbb{R}^2$

$f'_x(x_0, y_0) = \lim\limits_{h \to 0} \dfrac{f(x_0 + h, y_0) - f(x_0, y_0)}{h} = \dfrac{\partial f}{\partial x}(x_0, y_0) \qquad \underline{\underline{(1,0)}}$

$f'_y(x_0, y_0) = \lim\limits_{h \to 0} \dfrac{f(x_0, y_0 + h) - f(x_0, y_0)}{h} = \dfrac{\partial f}{\partial y}(x_0, y_0) \qquad \underline{\underline{(0,1)}}$

извод у правцу вектора $\vec{v} = (\alpha, \beta) \neq (0,0)$

$f'_{\vec{v}}(x_0, y_0) = \lim\limits_{h \to 0} \dfrac{f(x_0 + h \cdot \alpha, y_0 + h \cdot \beta) - f(x_0, y_0)}{h} = \dfrac{\partial f}{\partial \vec{v}}(x_0, y_0)$

$f:\mathbb{R}^2\to\mathbb{R}$ $\quad$ f диференцијабилна у (x_0,y_0) $\Leftrightarrow$

$$\Leftrightarrow \quad f(x_0+h,y_0+k) = f(x_0,y_0) + f'_x(x_0,y_0)\,h + f'_y(x_0,y_0)\cdot k + \underline{\underline{o}}\left(\sqrt{h^2+k^2}\right),\ (h,k)\to(0,0)$$

$$\Leftrightarrow \quad \lim_{(h,k)\to(0,0)} \frac{f(x_0+h,y_0+k)-f(x_0,y_0)-f'_x(x_0,y_0)\cdot h-f'_y(x_0,y_0)\cdot k}{\sqrt{h^2+k^2}} = 0$$

* f диф. у (x_0,y_0) $\Rightarrow$ f непр. у (x_0,y_0)
* f диф. у (x_0,y_0) $\Rightarrow$ $\exists\, f'_x(x_0,y_0), f'_y(x_0,y_0)$
*

$$\left.\begin{array}{l} f \text{ непр. у } (x_0,y_0) \\ \exists\, f'_x \text{ и } f'_y \text{ у некој околини } (x_0,y_0) \\ f'_x \text{ и } f'_y \text{ непр. у } (x_0,y_0) \end{array}\right\} \ \underline{\underline{f \text{ диф. у } (x_0,y_0)}}$$

* $f:\mathbb{R}^n\to\mathbb{R}$ $\quad x_1,\dots,x_n$

градијент: $\nabla f = (f'_{x_1}, f'_{x_2},\dots, f'_{x_n})$

$n=2$: $\nabla f(x_0,y_0) = (f'_x(x_0,y_0), f'_y(x_0,y_0))$

$$f'_{\vec{v}}(x_0,y_0) = (\nabla f(x_0,y_0), \vec{v})$$

1. $f(x,y,z) = 2y\sqrt{x} + 3y^2\sqrt[3]{z^2}$

Одредити парц. изводе

$$f'_x = 2y\cdot\frac{1}{2}\cdot\frac{1}{\sqrt{x}}$$

$$f'_y = 2\cdot\sqrt{x} + 3\cdot 2y\cdot\sqrt[3]{z^2}$$

$$f'_z = 3y^2\cdot\frac{2}{3}\cdot\frac{1}{\sqrt[3]{z}}$$

2. Док: да $z = y\cdot\ln(x^2-y^2)$ задовољава:

$$\frac{1}{x}z'_x + \frac{1}{y}\cdot z'_y = \frac{z}{y^2} \quad (*)$$

$$z'_x = y\cdot\frac{2x}{x^2-y^2} = \frac{2xy}{x^2-y^2}$$

$$z'_y = 1\cdot\ln(x^2-y^2) + y\cdot\frac{-2y}{x^2-y^2}$$

(*), лево $= \cancel{\dfrac{2y}{x^2-y^2}} + \dfrac{\ln(x^2-y^2)}{y} - \cancel{\dfrac{2y}{x^2-y^2}}$

$$= \frac{\ln(x^2-y^2)}{y}$$

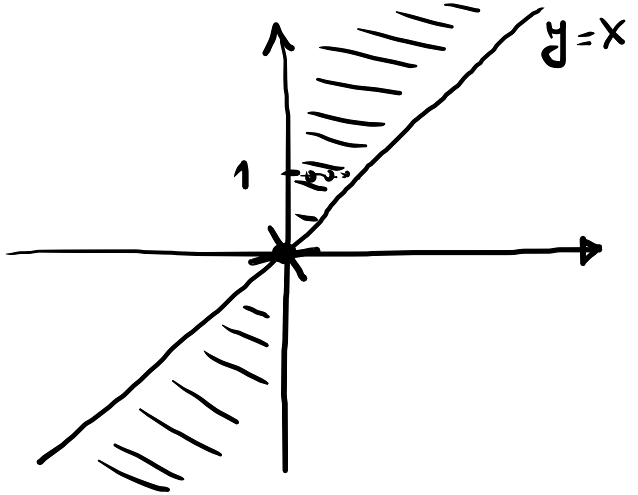
$$= \frac{z}{y^2}$$

③ $f(x,y) = x + (y-1)\cdot \arcsin\sqrt{\frac{x}{y}}$

$f'_x(x,1) = ?$, када $(x,1)$ припада домену f

домен: $\boxed{y \neq 0}$ $\quad \frac{x}{y} \geq 0 \quad\quad -1 \leq \sqrt{\frac{x}{y}} \leq 1$

$$\sqrt{\frac{x}{y}} \leq 1 \Rightarrow \boxed{0 \leq \frac{x}{y} \leq 1}$$



$$f'_x(x,1) = \lim_{h\to 0} \frac{f(x+h,1) - f(x,1)}{h} =$$

$$= \lim_{h\to 0} \frac{x+h + \cancel{(1-1)\arcsin\ldots} - x + 0}{h}$$

$$= \lim_{h\to 0} \underbrace{\frac{x+h-x}{h}}_{=1} = 1 \qquad \boxed{f'_x(x,1) = 1}$$

4. $f: \mathbb{R}^2 \to \mathbb{R}$

$$f(x,y) = \begin{cases} \dfrac{x \cdot y^3}{x^4+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

а) непр. f ?

б) Найти $f'_x(x,y), f'_y(x,y)$, ако $\exists$.

в) диф. f ?

а) f непр. на $\mathbb{R}^2 \setminus \{(0,0)\}$ ✓

$(0,0)$: $f(0,0) = 0 \overset{?}{=} \lim\limits_{(x,y)\to(0,0)} f(x,y)$

$$\lim_{(x,y)\to(0,0)} \frac{x \cdot y^3}{x^4+y^2} \overset{?}{=} 0$$

$$0 \leq \left| \frac{x \cdot y^3}{x^4+y^2} \right| \leq \left| \frac{x \cdot y^3}{y^2} \right| = |x \cdot y| \xrightarrow[(x,y)\to(0,0)]{} 0$$

$x^4+y^2 \geq y^2$

2-полицаја

$\Rightarrow \lim\limits_{(x,y)\to(0,0)} f(x,y) = 0$ непр. у $(0,0)$ ✓

$\Rightarrow f$ непр. на $\mathbb{R}^2$

б) $(x,y)\neq(0,0)$:

$$f'_x(x,y) = \frac{y^3(x^4+y^2) - x\cdot y^3\cdot 4x^3}{(x^4+y^2)^2} = \frac{y^5 - 3y^3x^4}{(x^4+y^2)^2}$$

$$f(x,y) = \begin{cases} \dfrac{xy^3}{x^4+y^2}, & (x,y)\neq(0,0) \\ 0, & (x,y)=(0,0) \end{cases}$$

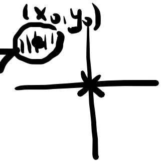
$$f'_y(x,y) = \frac{x\cdot 3y^2(x^4+y^2) - x\cdot y^3\cdot 2y}{(x^4+y^2)^2} = \frac{xy^4 + 3x^5y^2}{(x^4+y^2)^2}$$

$(0,0)$: $f'_x(0,0) = \lim_{h\to 0} \dfrac{f(h,0)-f(0,0)}{h} = \lim_{h\to 0}\dfrac{0-0}{h} = \boxed{0}$ $\qquad f'_y(0,0) = \boxed{0}$

в) диференцијабилност ?

$\mathbb{R}^2\setminus\{(0,0)\} \leftarrow$ f диференцијабилна ✓

($(x_0,y_0)\in\mathbb{R}^2\setminus\{(0,0)\}$ $\exists f'_x(x,y), f'_y(x,y)$ у околини (x_0,y_0) и непр. су. f непр.)



у $(0,0)$:

$$L = \lim_{(h,k)\to(0,0)} \frac{f(h,k) - \cancel{f(0,0)}^{0} - \cancel{f'_x(0,0)}^{0}\cdot h - \cancel{f'_y(0,0)}^{0}\cdot k}{\sqrt{h^2+k^2}} \overset{?}{=} 0$$

$$L = \lim_{(h,k)\to(0,0)} \frac{\frac{h\cdot k^3}{h^4+k^2}}{\sqrt{h^2+k^2}} = \lim_{(h,k)\to(0,0)} \frac{h\cdot k^3}{(h^4+k^2)\cdot\sqrt{h^2+k^2}} = 0 \quad ✓$$

2 полицајца:

$$0 \le \left|\frac{h\cdot k^3}{\underbrace{(h^4+k^2)}_{\ge k^2}\underbrace{\sqrt{h^2+k^2}}_{\ge\sqrt{k^2}=|k|}}\right| \le \left|\frac{h\cdot k^3}{k^2\cdot|k|}\right| = |h| \xrightarrow[(h,k)\to(0,0)]{} 0 \quad ✓$$

$L=0 \Rightarrow f$ диф. у $(0,0) \Rightarrow f$ диф. на $\mathbb{R}^2$

5. $f(x,y)=\begin{cases} \dfrac{x^2y}{\sqrt{x^4+y^4}}, & (x,y)\neq(0,0) \\ 0, & (x,y)=(0,0) \end{cases}$

а) непр. на $\mathbb{R}^2$?

б) извод у $\forall$ правцу у $(0,0)$?

в) да ли је f диф. у $(0,0)$?

(а) $\mathbb{R}^2\setminus\{(0,0)\}$ непр. ✓

$(0,0)$: $\underbrace{\lim_{(x,y)\to(0,0)} \frac{x^2y}{\sqrt{x^4+y^4}}}_{L} \overset{?}{=} 0$

$$0\leq\left|\frac{x^2y}{\sqrt{x^4+y^4}}\right| \underset{\sqrt{x^4+y^4}\geq x^2}{\leq} \left|\frac{x^2 y}{x^2}\right| = |y| \to 0, \quad (x,y)\to(0,0)$$

2-полицаја $\Rightarrow \boxed{L=0} \Rightarrow f$ непр. и у $(0,0)$ $\Rightarrow$ $\boxed{f \text{ непр. на } \mathbb{R}^2}$.

(б) $\vec{v}=(\alpha,\beta)\neq(0,0)$

$$\frac{\partial f}{\partial \vec{v}}(0,0)=\lim_{h\to0}\frac{f(h\alpha,h\beta)-f(0,0)}{h}=\lim_{h\to0}\frac{\frac{\cancel{h^2}\alpha^2\cancel{h}\beta}{\sqrt{\cancel{h^4}(\alpha^4+\beta^4)}}-0}{\cancel{h}}=\lim_{h\to0}\frac{\alpha^2\beta}{\sqrt{\alpha^4+\beta^4}}=\frac{\alpha^2\beta}{\sqrt{\alpha^4+\beta^4}}$$

$$\frac{\alpha^2\beta}{\sqrt{\alpha^4+\beta^4}} = \frac{\partial f}{\partial \vec{v}}(0,0)$$

(в) $\frac{\partial f}{\partial x}(0,0)=\frac{\partial f}{\partial \vec{v}}(0,0)=\mathbf{0}$, за $\vec{v}=(1,0)$, $\frac{\partial f}{\partial y}(0,0)=\frac{\partial f}{\partial \vec{u}}(0,0)=\mathbf{0}$, $\vec{u}=(0,1)$

$f'_x(0,0) = 0 = f'_y(0,0)$

f diff. u $(0,0)$ $\underline{\underline{?}}$:

$$L = \lim_{(h,k)\to(0,0)} \frac{f(h,k) - \cancelto{0}{f(0,0)} - \cancelto{0}{f'_x(0,0)}\, h - \cancelto{0}{f'_y(0,0)} \cdot k}{\sqrt{h^2+k^2}}$$

$$L = \lim_{(h,k)\to(0,0)} \underbrace{\frac{h^2 \cdot k}{\sqrt{h^4+k^4} \cdot \sqrt{h^2+k^2}}}_{g(h,k)}$$

$\underline{\underline{L \neq 0}}$: npr. $\underbrace{\left(\frac{1}{n}, \frac{1}{n}\right)}_{(h_n, k_n)} \to (0,0)$, $n \to \infty$

$$g(h_n,k_n) = \frac{\frac{1}{n^2} \cdot \frac{1}{n}}{\sqrt{2 \cdot \frac{1}{n^4}} \cdot \sqrt{2 \cdot \frac{1}{n^2}}} = \frac{\frac{1}{n^3}}{2 \cdot \frac{1}{n^3}} = \frac{1}{2} \not\to 0, \ n \to \infty$$

$\Rightarrow$ $\boxed{f \text{ nije diff u } (0,0)}$ $\square$

(6) $$f(x,y)=\begin{cases}(x^2+y^2)\cdot\sin\dfrac{1}{x^2+y^2}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}$$

Док. да f има прекидне парц. изводе у $(0,0)$, и неограничене, али да је ипак f диференцијабилна у $(0,0)$.

$$f'_x(0,0)=\lim_{h\to 0}\frac{f(h,0)-f(0,0)}{h}=\lim_{h\to 0}\frac{h^2\sin\frac{1}{h^2}}{h}=\lim_{h\to 0}h\cdot\underbrace{\sin\frac{1}{h^2}}_{\text{огр.}}=0 \qquad \boxed{f'_x(0,0)=0}$$

аналогно добијамо $\boxed{f'_y(0,0)=0}$

$$L=\lim_{(h,k)\to(0,0)}\frac{f(h,k)-\overset{f(0,0)}{0}-\overset{f'_x}{0}\cdot h-\overset{f'_y}{0}\cdot k}{\sqrt{h^2+k^2}}=\lim_{(h,k)\to(0,0)}\frac{(h^2+k^2)\cdot\sin\frac{1}{h^2+k^2}}{\sqrt{h^2+k^2}}$$

$$=\lim_{(h,k)\to(0,0)}\underbrace{\sqrt{h^2+k^2}\cdot\sin\frac{1}{h^2+k^2}}=0 \checkmark$$

$$0\leq|\cdot|\leq\sqrt{h^2+k^2}\to 0,\ (h,k)\to 0$$

$L=0 \Rightarrow$ $\boxed{f \text{ је диф у } (0,0)}$

$(x,y)\neq(0,0)$: $f'_x(x,y) = 2x\cdot\sin\frac{1}{x^2+y^2} + (x^2+y^2)\cdot\cos\frac{1}{x^2+y^2}\cdot\frac{-2x}{(x^2+y^2)^2}$ $\quad \left[(x^2+y^2)\sin\frac{1}{x^2+y^2}\right]$

$$f'_x(x,y) = 2x\cdot\sin\frac{1}{x^2+y^2} - \frac{2x}{x^2+y^2}\cdot\cos\frac{1}{x^2+y^2}$$

аналогно: $f'_y(x,y) = 2y\cdot\sin\frac{1}{x^2+y^2} - \frac{2y}{x^2+y^2}\cdot\cos\frac{1}{x^2+y^2}$

f'_x није непр. у $(0,0)$ $\iff \lim\limits_{(x,y)\to(0,0)} f'_x(x,y) \neq 0 = f'_x(0,0)$?

НИЗ: $(x_n,0)\to(0,0)$ „$\cos\frac{1}{x_n^2}=1$" „$x_n^2=\frac{1}{2n\pi}$" ← намештали

$$\boxed{\Big(\underbrace{\tfrac{1}{\sqrt{2n\pi}}}_{x_n},0\Big)\xrightarrow[n\to\infty]{}(0,0)}$$

$$f'_x(x_n,0) = 2x_n\cdot\cancel{\sin 2n\pi}^{\,0} - 2\cdot\frac{\frac{1}{\sqrt{2n\pi}}}{\frac{1}{2n\pi}}\cdot\underbrace{\cos 2n\pi}_{=1}$$

$$= -2\cdot\sqrt{2n\pi}\cdot 1$$

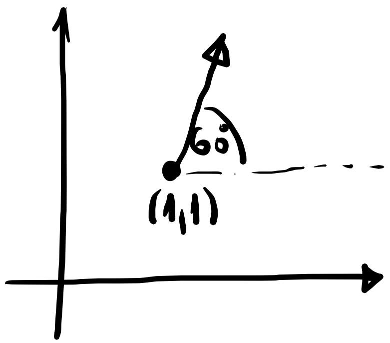
$$= -2\sqrt{2n\pi}\xrightarrow[n\to\infty]{} -\infty$$

$\Rightarrow \lim\limits_{(x,y)\to(0,0)} f'_x(x,y)\neq 0$ $\quad\Rightarrow f'_x$ није непр. у $(0,0)$

аналогно f'_y ш.

(7) $z(x,y)=x^2-y^2$

Наћи извод ф-је z у тачки $M(1,1)$ у правцу јединичног вектора који са позитивним делом x-осе гради угао од 60°.



$$\vec{v}=(\cos 60^\circ, \sin 60^\circ)=\boxed{\left(\frac{1}{2},\frac{\sqrt{3}}{2}\right)}$$

$$\frac{\partial f}{\partial \vec{v}}(x,y)=\left(\nabla f(x,y), \vec{v}\right)$$

$\nabla f = ?$ $\quad f=z$ ☺

$$f'_x=2x, \quad f'_y=-2y$$

$$f'_x(1,1)=2, \quad f'_y(1,1)=-2 \quad \Rightarrow \nabla f(1,1)=\boxed{(2,-2)}$$

$$\Rightarrow \frac{\partial f}{\partial \vec{v}}(x,y)=(2,-2)\circ\left(\frac{1}{2},\frac{\sqrt{3}}{2}\right)=\boxed{1-\sqrt{3}}$$

∎

За вежбу:

(*) $f(x,y) = \begin{cases} e^{\frac{x^3}{x^2+y^2}}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$

а) непр. f ?

б) диф f ?

испитати.