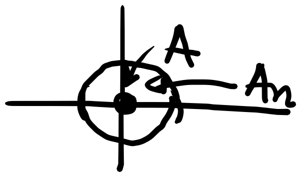
Основни појмови

Низови у $\mathbb{R}^n$

$A_n = (x_n, y_n, z_n) \in \mathbb{R}^3$, $n \in \mathbb{N}$ $\quad (A_n)_{n\in\mathbb{N}}$

Низ у $\mathbb{R}^k$: n-ти члан $A_n = (x_n^1, x_n^2, \dots, x_n^k)$



конвергенција (A_n) у $\mathbb{R}^k$, $A \in \mathbb{R}^k$

$$\lim_{n\to\infty} A_n = A \overset{\text{деф.}}{\iff} (\forall \varepsilon > 0)(\exists n_0 \in \mathbb{N})(\forall n > n_0)\ \underline{d(A_n, A)} < \varepsilon$$

$A_n = (x_n^1, \dots, x_n^k)$, $\quad A = (a^1, \dots, a^k)$

$$\lim_{n\to\infty} x_n^1 = a^1, \dots, \lim_{n\to\infty} x_n^k = a^k$$

„покоординатно"

1. $\lim_{n\to\infty} A_n = ?$ $\quad A_n = \left(\underbrace{\frac{2^n+3^n}{5\cdot 3^n-4}}_{x_n}, \underbrace{\left(1-\frac{1}{n}\right)^n}_{y_n}, \underbrace{\frac{\sin n}{n}}_{z_n}, \underbrace{n\cdot\sin\frac{1}{n}}_{t_n} \right)$

$$\lim_{n\to\infty} x_n \overset{:3^n}{=} \lim_{n\to\infty} \frac{\left(\frac{2}{3}\right)^n \xrightarrow{} 0 + 1}{5-\frac{4}{3^n} \to 0} = \frac{1}{5}$$

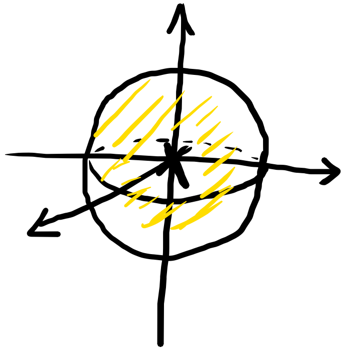
$$\lim_{n\to\infty}\left(1-\frac{1}{n}\right)^n = \lim_{n\to\infty}\left(\underbrace{\left(1+\frac{1}{\underset{\to -\infty}{-n}}\right)^{-n}}_{\to e}\right)^{-1} = \frac{1}{e}$$

$$\lim_{n\to\infty} \frac{\sin n}{\underset{\to\infty}{n}} = 0 \qquad \lim_{n\to\infty} n\cdot\sin\frac{1}{n} = \lim_{n\to\infty}\frac{\sin\frac{1}{n} \to 0}{\frac{1}{n}} = 1$$

$$\lim_{n\to\infty}(A_n) = \left(\frac{1}{5}, \frac{1}{e}, 0, 1\right)$$

Функције више променљивих

$f: \mathbb{R}^n \to \mathbb{R}$ домен D_f



1) $D_f = ?$

a) $f(x,y,z) = \dfrac{1}{\ln(1-x^2-y^2-z^2)}$ $f: \mathbb{R}^3 \to \mathbb{R}$

ln: $1-x^2-y^2-z^2 > 0$ $x^2+y^2+z^2 < 1$ · сфера, $R=1$, $(0,0,0)$

%: $1-x^2-y^2-z^2 \neq 1$ $x^2+y^2+z^2 \neq 0$ $(x,y,z) \neq (0,0,0)$

$(x-a)^2+(y-b)^2+(z-c)^2$

унут. $< r^2$

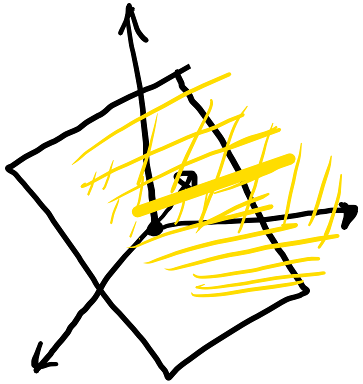
спољ. $> r^2$

сфера $= r^2$

б) $f(x,y,z) = \sqrt[4]{\underbrace{x+y+z}_{\geq 0}}$ $f: \mathbb{R}^3 \to \mathbb{R}$

$x+y+z \geq 0$ – полупростор

$x+y+z = 0$ – раван $(1,1,1)$

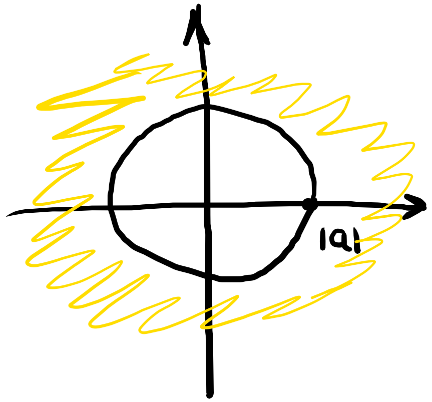


$ax+by+cz=d$

(a,b,c)

в) $f: \mathbb{R}^2 \to \mathbb{R}$ $\quad f(x,y) = \dfrac{1}{\sqrt{x^2+y^2-a^2}}$

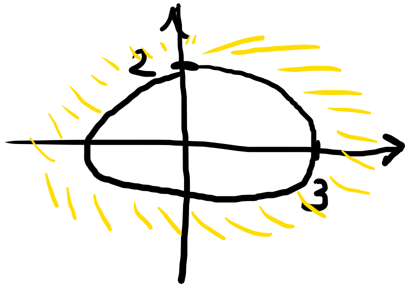
$$x^2+y^2-a^2>0$$
$$x^2+y^2>a^2$$
$$(=)$$



* $f(x,y) = \dfrac{1}{\sqrt{4x^2+9y^2-36}}$ $\quad D_f = ?$

$4x^2+9y^2-36>0 \qquad 4x^2+9y^2>36 \ / :36$

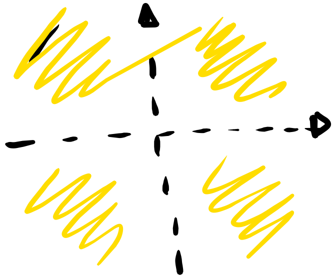
$\dfrac{x^2}{9}+\dfrac{y^2}{4}>1$ спољ. елипсе



г) $f: \mathbb{R}^2 \to \mathbb{R}$

$f(x,y) = \dfrac{1}{x^5 \cdot y^{2021}}$ $\quad D_f = ?$

$$x^5 \cdot y^{2021} \neq 0$$
$$x \neq 0, y \neq 0$$



$$D_f = \mathbb{R}^2 \setminus \{(x,y)\in\mathbb{R}^2 \mid x=0 \vee y=0\}$$

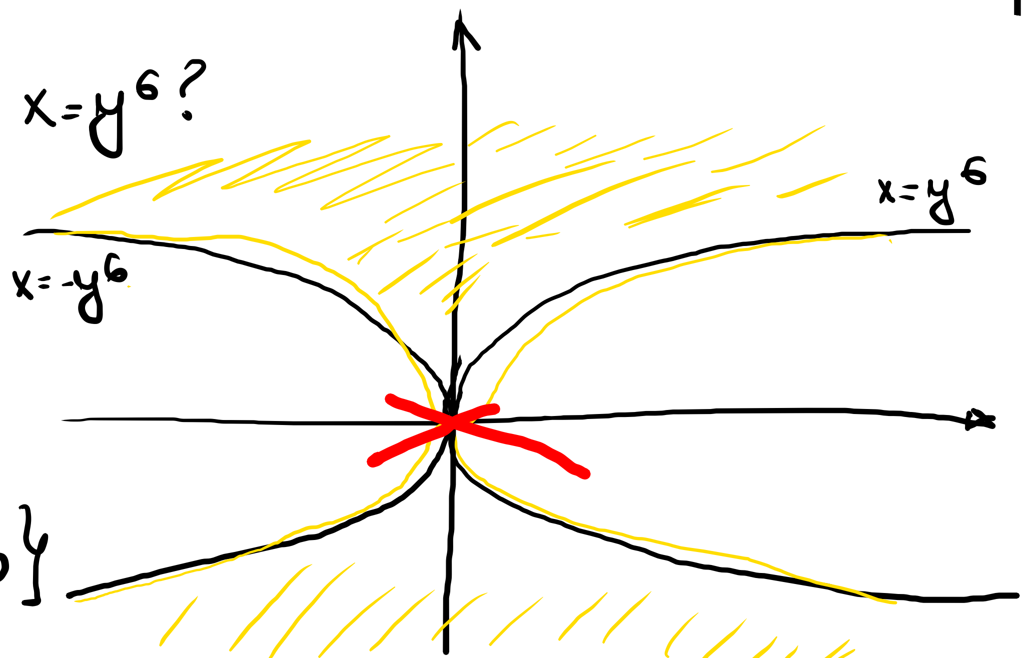
2. $D_f = ?$

a) $f : \mathbb{R}^2 \to \mathbb{R}$ $\quad f(x,y) = \arcsin \dfrac{x}{y^6}$

$\underline{y \neq 0}$ $\quad -1 \leq \dfrac{x}{y^6} \leq 1 \quad / \cdot y^6, \; y^6 \geq 0$

$\Leftrightarrow \quad \underline{-y^6 \leq x \leq y^6}$

$$D_f = \{(x,y) \in \mathbb{R}^2 \mid -y^6 \leq x \leq y^6, \; y \neq 0\}$$

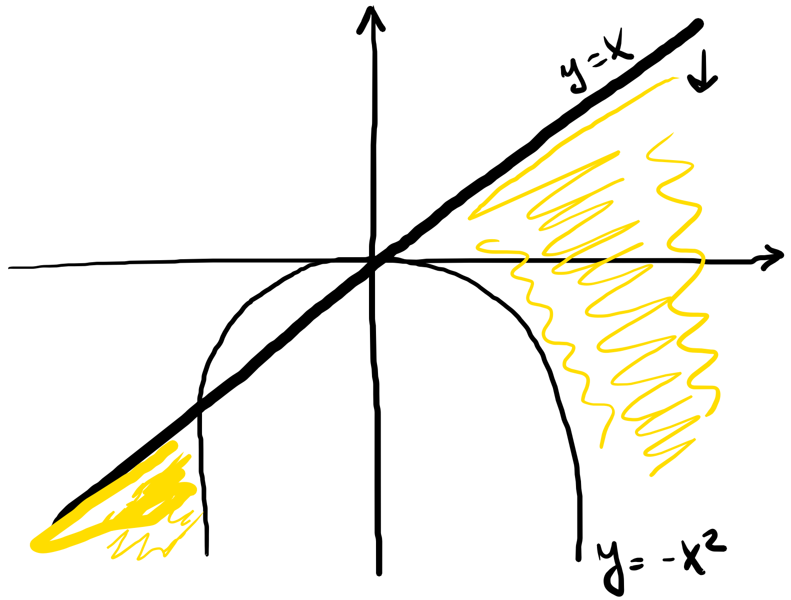


б) $f : \mathbb{R}^2 \to \mathbb{R}$ $\quad f(x,y) = \dfrac{\ln(x-y) - 2}{\sqrt{x^2+y}}$

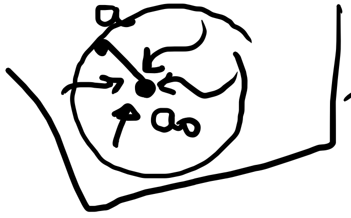
$x - y > 0 \quad \underline{y < x}$

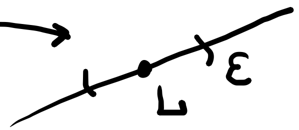
$x^2 + y > 0 \quad y > \underline{-x^2}$

$\boxed{-x^2 < y < x}$



Гранична вредност





ДЕФ $f:\mathbb{R}^n\to\mathbb{R}$ $f(a)$ a_0 – тачка нагомилавања D_f

$$\lim_{a\to a_0} f(a)=L \overset{\text{деф.}}{\Longleftrightarrow} (\forall\varepsilon>0)(\exists\delta>0)(\forall a\in D_f)\ \ 0<d(a,a_0)<\delta \Rightarrow |f(a)-L|<\varepsilon$$

$\mathbb{R}^2$

$f:\mathbb{R}^2\to\mathbb{R}$ $f(x,y)$

$\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y)$ ДВОЈНИ ЛИМЕС

ПОНОВЉЕНИ ЛИМЕСИ

$\lim\limits_{x\to 0}\lim\limits_{y\to 0} f(x,y)$

$\lim\limits_{y\to 0}\lim\limits_{x\to 0} f(x,y)$

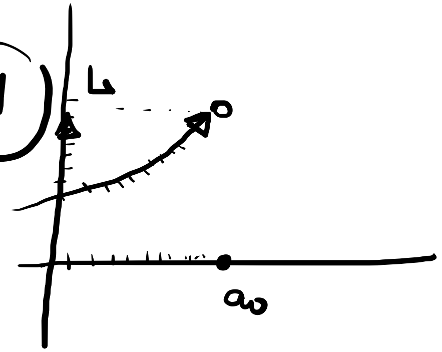
могу се разликовати!

1) $f: D_f \to \mathbb{R}$ $\quad f(x,y) = \dfrac{x-y}{x+y}$

а) поновљени лимеси? $\lim\limits_{x\to 0}\lim\limits_{y\to 0} f(x,y)$, $\lim\limits_{y\to 0}\lim\limits_{x\to 0} f(x,y)$

б) док. да $\neg\exists \lim\limits_{(x,y)\to(0,0)} f(x,y)$

а) $$\lim_{x\to 0}\lim_{y\to 0}\underbrace{\frac{x-y}{x+y}}_{\to 1} = \lim_{x\to 0} 1 = 1 \qquad \lim_{y\to 0}\lim_{x\to 0}\underbrace{\frac{x-y}{x+y}}_{\to -1} = \lim_{y\to 0}(-1) = -1$$



б) $\neg\exists$

Хајнеова т. $f: \mathbb{R}^n \to \mathbb{R}$

$\lim\limits_{a\to a_0} f(a) = L \iff$ за $\forall$ низ $a_n \to a_0$ важи $\lim\limits_{n\to\infty} f(a_n) = L$

$a_n = \left(\frac{1}{n}, \frac{1}{n}\right) \xrightarrow[n\to\infty]{} (0,0)$ $\qquad f(a_n) = 0 \xrightarrow[n\to\infty]{} \boxed{0}$

$b_n = \left(\frac{2}{n}, \frac{1}{n}\right) \to (0,0)$ $\qquad f(b_n) = \dfrac{1/n}{3/n} = \dfrac{1}{3} \xrightarrow[n\to\infty]{} \boxed{1/3}$

$\ast$

↳ Хајне

$\Rightarrow \neg\exists \lim\limits_{(x,y)\to(0,0)} f(x,y)$

② $f(x,y) = \dfrac{x^2y^2}{x^2y^2+(x-y)^2}$

а) док. да ∃ повторни лимеси $x\to0, y\to0$, и $y\to0, x\to0$

б) док. да ∄ $\lim\limits_{(x,y)\to(0,0)} f(x,y)$

а) $$\lim_{x\to0}\lim_{y\to0}\frac{\overset{\to 0}{x^2y^2}}{\underbrace{\underset{0}{x^2y^2}+\underset{\to x^2}{(x-y)^2}}_{\text{"}\frac{0}{x^2}\text{"}\to 0}} = \lim_{x\to0} 0 = \boxed{0}$$

аналогично: $\lim\limits_{y\to0}\lim\limits_{x\to0} f(x,y) = \boxed{0}$

б) $a_n = \left(\frac1n, \frac1n\right) \xrightarrow[n\to\infty]{} (0,0)$ $\qquad f(a_n) = \dfrac{\frac{1}{n^4}}{\frac{1}{n^4}+0} = 1 \xrightarrow[n\to\infty]{} 1$

$b_n = \left(\frac1n, -\frac1n\right) \xrightarrow[n\to\infty]{} (0,0)$ $\qquad f(b_n) = \dfrac{\frac{1}{n^4}}{\frac{1}{n^4}+\frac{4}{n^2}} \overset{\cdot n^4}{=} \dfrac{1}{1+4n^2} \xrightarrow[n\to\infty]{} 0$

Хайне $\Rightarrow$ ∄ $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ □