Још 1 задатак из криволинијских интеграла:

20.5.2021.

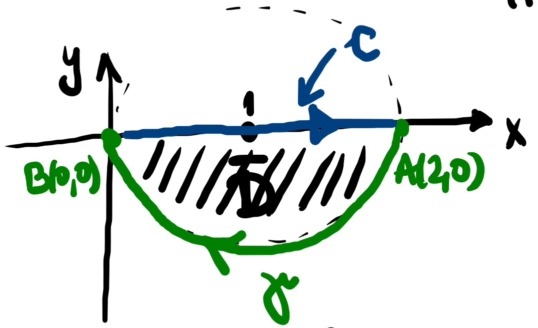
$\boxed{1.}$ $I = \int_\gamma F\cdot dr$, $\gamma: y = -\sqrt{2x-x^2}$ од $A(2,0)$ до $B(0,0)$

$F(x,y) = (\underbrace{e^{x+y}\cdot \sin 2y + x+y}_{P},\ \underbrace{e^{x+y}(2\cos 2y + \sin 2y) + 2x}_{Q})$

$\gamma: \quad y = -\sqrt{2x-x^2} \iff y^2 = 2x - x^2 \wedge y \le 0$

$\iff y^2 + (x-1)^2 = 1 \wedge y \le 0$

$(1,0)$ $r=1$



$Q'_x - P'_y \overset{...}{=}$ лепо

волели бисмо Гринову формулу

γ - није затворена $\to$ $\gamma \cup c$ затв.

$\boxed{\gamma\cup c}$: P, Q - непр ✓

непр ✓ $P'_y(x,y) = e^{x+y}\cos 2y \cdot 2 + e^{x+y}\cdot \sin 2y + 1$

$Q'_x(x,y) = e^{x+y}\cdot(2\cos 2y + \sin 2y) + 2$

$\boxed{Q'_x - P'_y = 1}$

$\gamma\cup c$ није поз. орј. $\Rightarrow$ $-$

$\boxed{\oint_{\gamma\cup c} F\cdot dr} \overset{\text{Грин}}{=} -\iint_{\bar D} (Q'_x - P'_y)\,dx\,dy = \iint_{\bar D} 1\,dx\,dy = -P(\bar D)$

$= -\frac{1}{2} r^2 \pi = \boxed{\frac{-\pi}{2}}$

$\overbrace{\oint_{\gamma\cup c} F\cdot dr}^{-\pi/2} = \overbrace{\int_\gamma F\cdot dr}^{I} + \int_c F\cdot dr$ ②

$\boxed{c}$ $r(t) = (t,0)$, $t\in[0,2]$

$\boxed{\int_c F\cdot dr} = \int_0^2 F(r(t))\cdot r'(t)\,dt = \int_0^2 (t, Q(t,0))\cdot(1,0)\,dt$

$= \int_0^2 t\,dt = \frac{t^2}{2}\Big|_0^2 = \boxed{2}$

$I = \oint_{\gamma\cup c} - \int_c = \boxed{\frac{-\pi}{2} - 2}$ ◻

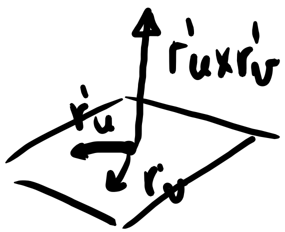
~ Површински интеграл I врсте ~

$S \subset \mathbb{R}^3$ површ: $r: \bar{D} \to \mathbb{R}^3$ (услови) $\boxed{r(u,v)} = (x(u,v), y(u,v), z(u,v))$ – ПАРАМЕТРИЗАЦИЈА

r регуларна: r'_u и r'_v лин. нез. на D

• $r'_u \times r'_v$:

$$\begin{vmatrix} i & j & k \\ & r'_u & \\ & r'_v & \end{vmatrix}$$



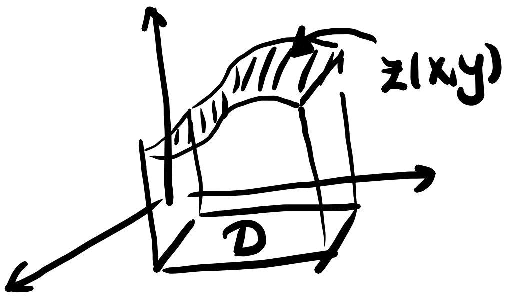
ПОВРШИНСКИ ИНТЕГРАЛ

$f: S \to \mathbb{R}$ S-површ

$$\iint_S f\,dS = \iint_{\bar{D}} f(r(u,v)) \cdot \underbrace{\|r'_u \times r'_v\|\, du\, dv}_{dS \text{ - елемент површине}}$$

површина S: $P(S) = \iint_{\bar{D}} \|r'_u \times r'_v\|\, du\, dv$

☺ ако је S график функције:



$r(x,y) = (x, y, z(x,y)), \ (x,y) \in D$

$r'_x = (1, 0, z'_x)$

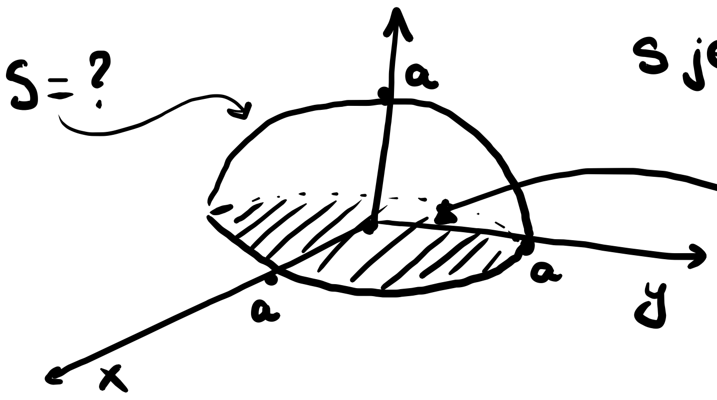
$r'_y = (0, 1, z'_y)$

$$r'_x \times r'_y = \begin{vmatrix} i & j & k \\ 1 & 0 & z'_x \\ 0 & 1 & z'_y \end{vmatrix} = (-z'_x, -z'_y, 1)$$

$$\|r'_x \times r'_y\| = \sqrt{1 + (z'_x)^2 + (z'_y)^2}$$

$$dS = \sqrt{1 + (z'_x)^2 + (z'_y)^2}\, dx\, dy$$

1. $I=\iint\limits_S (x+y+z)\,dS \qquad S=\{(x,y,z)\in\mathbb{R}^3 \mid x^2+y^2+z^2=a^2,\ z\geq 0\} \quad (a>0)$



S je полусфера $(z\geq 0)$ центар $(0,0,0)$, полупр. $=a$

$D=$ пројј. S на xy-раван

$D=\{(x,y)\in\mathbb{R}^2 \mid x^2+y^2\leq a^2\}$

S = график фје $\boxed{z(x,y)=\sqrt{a^2-x^2-y^2}}$

$dS=?$ ☺ S-график $\Rightarrow dS=\sqrt{1+(z'_x)^2+(z'_y)^2}\,dxdy$

$z'_x=\frac{1}{2}\frac{-2x}{\sqrt{a^2-x^2-y^2}}=\frac{-x}{\sqrt{a^2-x^2-y^2}}$

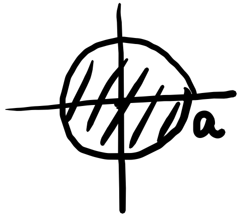
$z'_y=\frac{-y}{\sqrt{a^2-x^2-y^2}}$

$$dS=\sqrt{1+\frac{x^2}{a^2-x^2-y^2}+\frac{y^2}{a^2-x^2-y^2}}\,dxdy$$

$$dS=\sqrt{\frac{a^2}{a^2-x^2-y^2}}\,dxdy \overset{a>0}{=} \boxed{\frac{a}{\sqrt{a^2-x^2-y^2}}\,dxdy}$$

$$\Rightarrow I=\iint\limits_S (x+y+z)\,dS=\iint\limits_D \overbrace{\left(x+y+\sqrt{a^2-x^2-y^2}\right)}^{f}\cdot\frac{a}{\sqrt{a^2-x^2-y^2}}\,dxdy=$$

$$\Rightarrow I = \iint_D \left(\frac{ax}{\sqrt{a^2-x^2-y^2}} + \frac{ay}{\sqrt{a^2-x^2-y^2}} + a \right) dx\,dy$$



$$D: \quad x=\rho\cos\theta \quad \rho\in[0,a] \qquad J=\rho \qquad x^2+y^2=\rho^2$$
$$\quad\;\; y=\rho\sin\theta \quad \theta\in[-\pi,\pi]$$

$$\Rightarrow I = \int_0^a \left(\int_{-\pi}^{\pi} \left(\frac{a\cdot\rho\cos\theta}{\sqrt{a^2-\rho^2}} + \frac{a\cdot\rho\sin\theta}{\sqrt{a^2-\rho^2}} + a \right)\rho\, d\theta \right) d\rho$$

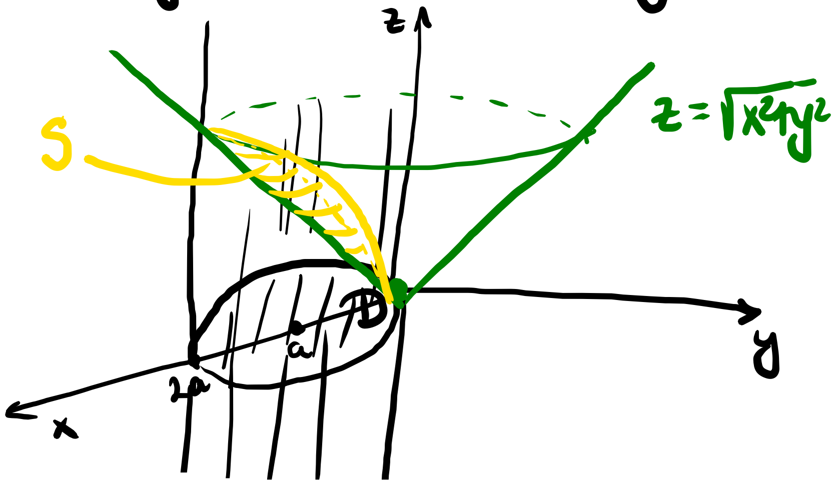
$$= \int_0^a \left(\underbrace{\cancel{\frac{a\rho^2}{\sqrt{a^2-\rho^2}} \cdot \underbrace{\int_{-\pi}^{\pi} (\cos\theta+\sin\theta)\,d\theta}_{=(\sin\theta-\cos\theta)\big|_{-\pi}^{\pi} \;=0}}}_{0} + a\rho \underbrace{\int_{-\pi}^{\pi} d\theta}_{2\pi} \right) d\rho$$

$$= \int_0^a a\rho\cdot 2\pi\, d\rho = \sqcup$$

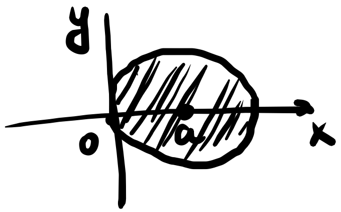
$$= \boxed{a^3\pi}$$

2. $I = \iint\limits_S (xy+yz+zx)\,dS$ S = део конуса $z=\sqrt{x^2+y^2}$ одсечен цилиндром $x^2+y^2=2ax$ $(a>0)$

$x^2+y^2=2ax \iff (x-a)^2+y^2=a^2$



D:



$\boxed{D}$ ✓

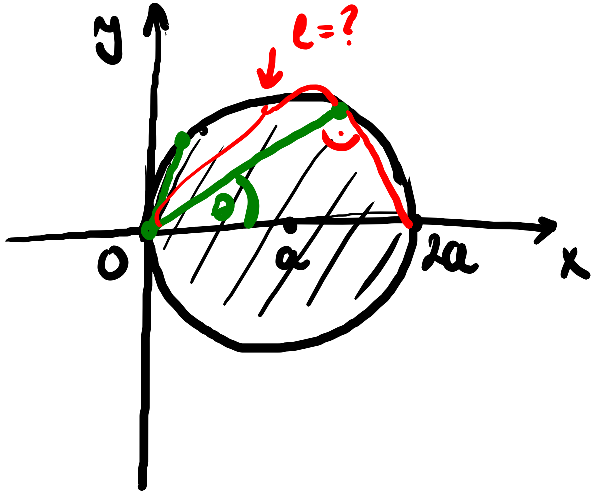
$\boxed{\Gamma}$: $r(x,y)=(x,y,z(x,y))=(x,y,\sqrt{x^2+y^2}),\ (x,y)\in D$

$dS=\sqrt{1+(z_x')^2+(z_y')^2}\,dxdy=\sqrt{1+\underbrace{\left(\frac{x}{\sqrt{x^2+y^2}}\right)^2+\left(\frac{y}{\sqrt{x^2+y^2}}\right)^2}_{=1}}\,dxdy$

$\boxed{dS=\sqrt{2}\,dxdy}$

$I=\iint\limits_S (xy+yz+zx)\,dS=\iint\limits_D \left(xy+y\sqrt{x^2+y^2}+x\sqrt{x^2+y^2}\right)\sqrt{2}\,dxdy$

D → померене поларне коорд ? не, штавише, боље обичне :)



$x = \rho\cos\theta$

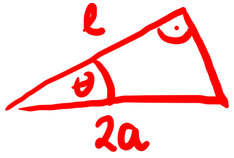
$y = \rho\sin\theta$

$J = \underline{\underline{\rho}} \qquad \sqrt{x^2+y^2} = \sqrt{\rho^2} \overset{\rho>0}{=} \underline{\underline{\rho}}$

$\boxed{\text{Границе за } \theta \text{ и } \rho}$

$\boxed{\theta} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \qquad \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$\rho \in \ ?$



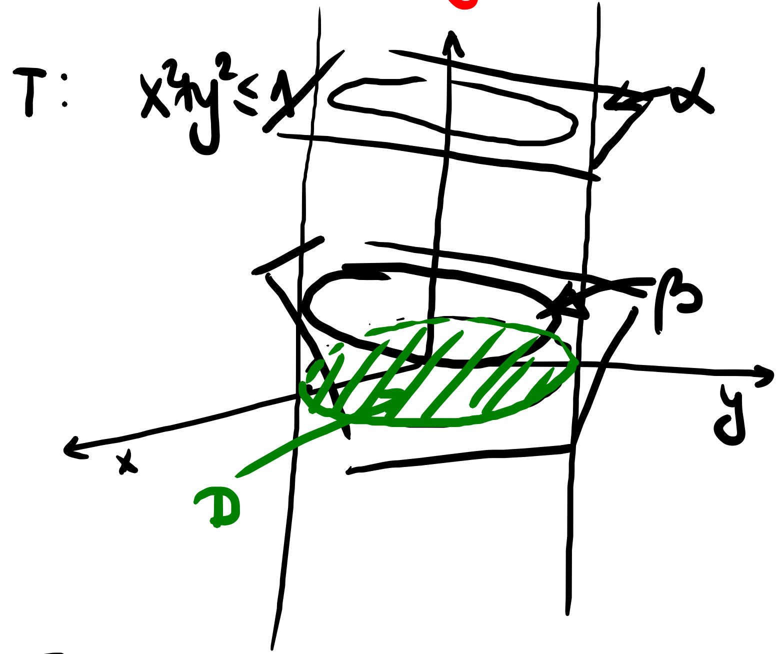
$l = 2a\cdot\cos\theta$

$\Rightarrow \boxed{\rho} \in [0, 2a\cdot\cos\theta]$

$$\Rightarrow I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\int_0^{2a\cos\theta} \Big(\overbrace{\rho^2\cos\theta\sin\theta + \rho\cdot(\cos\theta+\sin\theta)\cdot\rho}^{\text{ф-ја}}\Big)\cdot \overset{J}{\rho} \cdot \sqrt{2}\, d\rho \right) d\theta$$

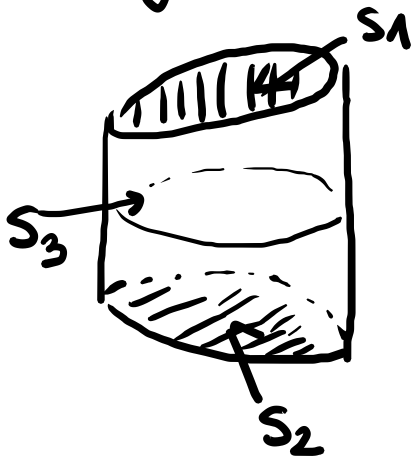
$$\overset{\dots}{=} \boxed{\frac{64\sqrt{2}}{15}} \quad \square$$

3 Одредити површину границе тела $T=\{(x,y,z)\in\mathbb{R}^3 \mid x^2+y^2\le 1,\ 3x+2y+z\le 6,\ x+y+z\ge 1\}$



$3x+2y+z\le 6 \leftarrow \alpha:\ 3x+2y+z=6$

$x+y+z\ge 1 \leftarrow \beta:\ x+y+z=1$



$$\boxed{P(\partial T)=P(S_1)+P(S_2)+P(S_3)}$$

$\boxed{S_1}$ $3x+2y+z=6$

$\boxed{z(x,y)}=6-3x-2y$

$(x,y)\in ?$ $\boxed{(x,y)\in D}$ – пројекција унутр. цилиндра

$D=\{(x,y)\in\mathbb{R}^2 \mid x^2+y^2\le 1\}$

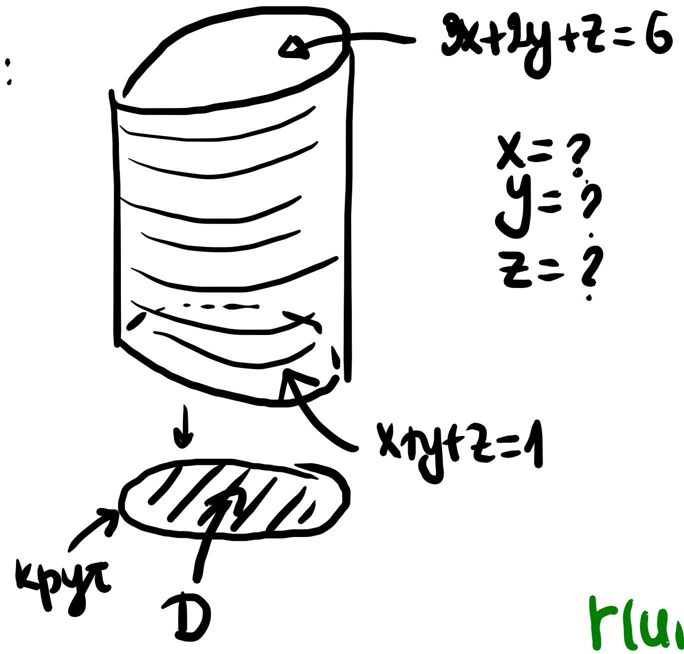
$$P(S_1)=\iint_D dS = \iint_D \sqrt{1+(z'_x)^2+(z'_y)^2}\,dx\,dy$$

$$=\iint_D \sqrt{1+(-3)^2+(-2)^2}\,dx\,dy=\sqrt{14}\iint_D dx\,dy$$

$$=\sqrt{14}\cdot P(D)=\sqrt{14}\cdot 1^2\pi=\boxed{\sqrt{14}\pi}$$

$\boxed{S_2}$ аналогно: део β: $x+y+z=1$ $\boxed{z(x,y)=1-x-y}$ $(x,y)\in D$ $\leadsto$ $\boxed{P(S_2)=\sqrt{3}\cdot\pi}$

S_3:



$x=?$
$y=?$
$z=?$

$\left.\begin{array}{l} x=\cos u \\ y=\sin u \end{array}\right\}$ „(x,y) иде по кругу"

$u\in[-\pi,\pi]$ → задаје D_1 !

$z=v$

$v\in[\underbrace{1-\cos u-\sin u}_{1-x-y},\ \underbrace{6-3\cos u-2\sin u}_{6-3x-2y}]$

$r(u,v)=(\cos u,\sin u,v)$ $\quad (u,v)\in D_1$

$r'_u=(-\sin u,\cos u,0)$ $\quad r'_v=(0,0,1)$

$$\boxed{r'_u\times r'_v}=\begin{vmatrix} i & j & k \\ -\sin u & \cos u & 0 \\ 0 & 0 & 1 \end{vmatrix}=(\cos u,\sin u,0)$$

$$\boxed{\|r'_u\times r'_v\|}=\sqrt{\cos^2u+\sin^2u+0}=1$$

$$P(S_3)=\iint_{S_3}dS=\iint_{D_1}\underbrace{\|r'_u\times r'_v\|}_{1}\,du\,dv=\int_{-\pi}^{\pi}\left(\int_{1-\cos u-\sin u}^{6-3\cos u-2\sin u}1\,dv\right)du=\int_{-\pi}^{\pi}(6-3\cos u-2\sin u)-(1-\cos u-\sin u)\,du$$

$$=\int_{-\pi}^{\pi}(5-2\cos u-\sin u)\,du=\boxed{10\pi}$$

$$P(S)=P(S_1)+P(S_2)+P(S_3)=(\sqrt{14}+\sqrt{3}+10)\pi \quad \square$$

3a beh*y : $P(S) = ?$ $S = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2 = R^2\}$

$x(\theta,\varphi) = R\sin\varphi\cos\theta$ $\theta \in [-\pi,\pi)$

$y(\theta,\varphi) = R\sin\varphi\sin\theta$ $\varphi \in [0,\pi]$

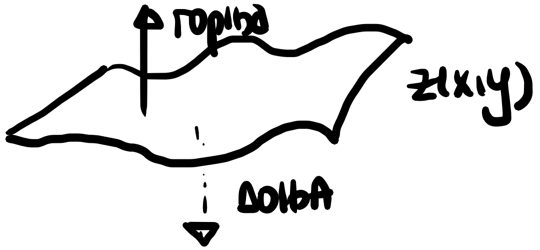
$z(\theta,\varphi) = R\cos\varphi$

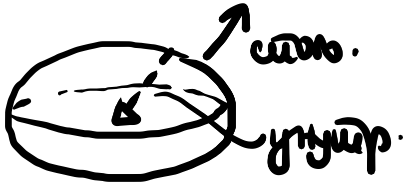
$r(\theta,\varphi) = (x(\theta,\varphi), y(\theta,\varphi), z(\theta,\varphi))$

r'_θ, r'_φ $\|r'_\theta \times r'_\varphi\| = \dots$

...

~ Површински интеграли II врсте ~

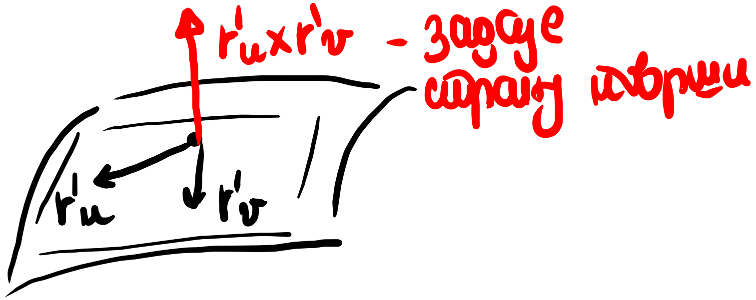




рег. парам. $r: D \to \mathbb{R}^3$ површи S $\quad r(u,v)$

↓

оријентацију: $r'_u, r'_v \quad r'_u \times r'_v$



векторско поље $F: S \to \mathbb{R}^3$ на S

$F = (P, Q, R)$

$$\underbrace{\iint_S F \cdot d\vec{S}}_{\iint_S P\,dydz + Q\,dzdx + R\,dxdy} := \iint_S F \underset{\text{скал. пр.}}{\circ} \vec{n}\, dS, \qquad \vec{n} = \frac{r'_u \times r'_v}{\|r'_u \times r'_v\|}$$

Како рачунамо?

$r: D \to \mathbb{R}^3$ рег. парам. на S

S - орј. сагласно са параметризацијом

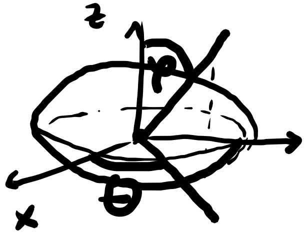
F - непр. вект. поље

$$\iint_S F \cdot d\vec{S} = \iint_D F(r(u,v)) \bullet (r'_u \times r'_v)\, du\, dv$$

⊛ S, S^- : $\iint_{S^-} F \cdot d\vec{S} = -\iint_{S} F \cdot d\vec{S}$

$\boxed{1.}$ $I = \iint_S F \cdot d\vec{S}$ S - спољашња страна елипсоида: $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$, $F(x,y,z) = \left(\frac{1}{x}, \frac{1}{y}, \frac{1}{z}\right)$.

$\boxed{S}$ $\underset{\sim}{r}$



$x = a \cdot \sin\varphi \cos\theta$

$y = b \cdot \sin\varphi \sin\theta$

$z = c \cdot \cos\varphi$

$\boxed{\begin{matrix}\varphi \in [0,\pi] \\ \theta \in [-\pi,\pi]\end{matrix}}$ ← задаје D

$\boxed{r(\theta,\varphi)} = (a \sin\varphi\cos\theta, b \sin\varphi \sin\theta, c\cos\varphi)$

$r'_\theta = ?$

$r'_\varphi = ?$

$r(\theta,\varphi) = (a\sin\varphi\cos\theta,\ b\sin\varphi\sin\theta,\ c\cos\varphi)$

$r'_\theta = (-a\sin\varphi\sin\theta,\ b\sin\varphi\cos\theta,\ 0)$

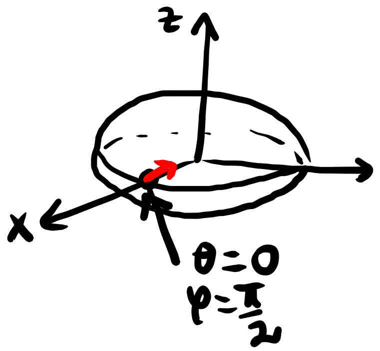
$r'_\varphi = (a\cos\varphi\cos\theta,\ b\cos\varphi\sin\theta,\ -c\sin\varphi)$

$$\boxed{r'_\theta \times r'_\varphi} = \begin{vmatrix} i & j & k \\ -a\sin\varphi\sin\theta & b\sin\varphi\cos\theta & 0 \\ a\cos\varphi\cos\theta & b\cos\varphi\sin\theta & -c\sin\varphi \end{vmatrix} = \dots = \underbrace{(-bc\cos\theta\cdot\sin^2\varphi,\ -ac\sin\theta\sin^2\varphi,\ -ab\sin\varphi\cos\varphi)}_{(*)}$$

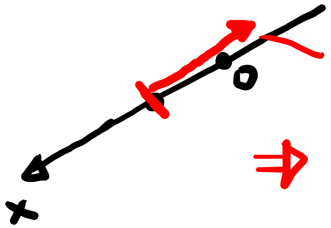
да ли је S орј. сагласно са r ?

S: спољашња страна

r: $r'_\theta \times r'_\varphi$ 1 тачка: $(\theta,\varphi) = (0,\frac{\pi}{2})$



$r'_\theta \times r'_\varphi(0,\frac{\pi}{2}) = (\underbrace{-bc}_{<0},\ 0,0)$ $(b,c>0)$



ка унутрашњости

⇒ S и r нису сагласне

$$I = \iint_S F\cdot d\vec{S}$$

$$= -\iint_D F(r(\theta,\varphi))\cdot(r'_\theta \times r'_\varphi)\,d\theta\,d\varphi$$

$$= -\int_0^{\pi}\int_{-\pi}^{\pi}\left(\frac{1}{a\cos\theta\cdot\sin\varphi},\ \frac{1}{b\sin\varphi\sin\theta},\ \frac{1}{c\cos\varphi}\right)\cdot (*)\ d\theta\,d\varphi$$

$$\overset{\dots}{=} -\int_0^{\pi}\int_{-\pi}^{\pi}\left(-\frac{bc}{a}\sin\varphi + -\frac{ac}{b}\sin\varphi + -\frac{ab}{c}\sin\varphi\right)d\theta\,d\varphi$$

$$\overset{\dots}{=} 4\pi\cdot\left(\frac{bc}{a}+\frac{ac}{b}+\frac{ab}{c}\right) \quad \square$$