~ Диференцијабилност ~

прво - подсетање - теорија pdf

① $f' = ?$

(a) $f(x) = x^3 \cdot e^x \cdot \ln x$

$$f'(x) = (\underbrace{x^3}_{f_1} \cdot \underbrace{(e^x \cdot \ln x)}_{f_2})'$$

$$= (x^3)' \cdot (e^x \cdot \ln x) + x^3 \cdot (e^x \cdot \ln x)'$$

$$= 3 \cdot x^2 \cdot e^x \cdot \ln x + x^3 \cdot (\underbrace{(e^x)'}_{e^x} \ln x + e^x \cdot \underbrace{(\ln x)'}_{1/x})$$

$$= 3x^2 \cdot e^x \cdot \ln x + x^3 \cdot e^x \cdot \ln x + x^3 \cdot e^x \cdot \frac{1}{x}$$

$$= \ldots$$

$$(f_1 \cdot f_2)' = f_1' \cdot f_2 + f_1 \cdot f_2'$$

$$(f_1 \cdot f_2 \cdot f_3)' = f_1' f_2 f_3 + f_1 \cdot f_2' f_3 + f_1 \cdot f_2 \cdot f_3'$$

$\vdots$

индукција:

$$(f_1 \cdots f_n)' = f_1' \cdot f_2 \cdots f_n + f_1 \cdot f_2' \cdot f_3 \cdots f_n + \ldots + f_1 \cdots f_{n-1} \cdot f_n'$$

(δ) $f(x)=\sqrt{1+\sqrt[3]{1+x^4}}$

$$\leftarrow \left(\sqrt{g(x)}\right)' = \overbrace{\frac{1}{2}\cdot\frac{1}{\sqrt{g(x)}}}^{\text{υξωgr}}\cdot g'(x)$$

$$f'(x)=\frac{1}{2}\cdot\frac{1}{\sqrt{1+\sqrt[3]{1+x^4}}}\cdot\Big(1+\underbrace{\sqrt[3]{1+x^4}}_{=(1+x^4)^{1/3}}\Big)'$$

$$=\frac{1}{2}\cdot\frac{1}{\sqrt{1+\sqrt[3]{1+x^4}}}\cdot\Big(0+\frac{1}{3}\cdot(1+x^4)^{-\frac{2}{3}}\cdot\underbrace{(1+x^4)'}_{0+4x^3}\Big)$$

$$=\frac{1}{2}\cdot\frac{1}{\sqrt{1+\sqrt[3]{1+x^4}}}\cdot\frac{1}{3}\cdot\frac{1}{(1+x^4)^{2/3}}\cdot 4\cdot x^3$$

$$\sqrt{t}=t^{1/2}$$

$$(\sqrt{t})'=\frac{1}{2}\cdot t^{1-\frac{1}{2}}=\frac{1}{2}\cdot t^{-1/2}$$

$$=\frac{1}{2\sqrt{t}}$$

(6) $f(x)=x^x$

$$f'(x)=\left(e^{x\cdot\ln x}\right)'=e^{x\ln x}\cdot(x\cdot\ln x)'$$

$$=e^{x\cdot\ln x}\cdot\left(1\cdot\ln x+x\cdot\frac{1}{x}\right)$$

$$=x^x\cdot(\ln x+1)$$

☺ $\left(u(x)^{w(x)}\right)'=\left(e^{w(x)\cdot\ln(u(x))}\right)'$

$=\dots$

г) $f(x)=x^{x^x}$

$$f'(x)=\left(e^{x^x\cdot \ln x}\right)' = e^{x^x \ln x}\cdot\left(x^x \cdot \ln x\right)'$$

$$= e^{x^x \ln x}\cdot\Big(\underbrace{(x^x)'}_{x^x(\ln x+1)\ \underline{\underline{\text{б}}}}\cdot \ln x + x^x\cdot \frac{1}{x}\Big)$$

$$= x^{x^x}\cdot\left(x^x(\ln x+1)\cdot \ln x + x^x\cdot\frac{1}{x}\right)$$

$$= x^{x^x}\cdot x^x\left((\ln x)^2+\ln x+\frac{1}{x}\right)$$

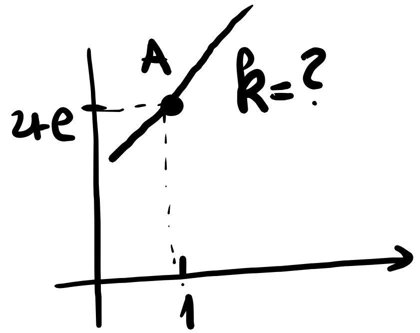
За вежбу:

* $f(x)=(\operatorname{arctg} x)^{x^2}$ $\qquad f'(x)=?$

* $f(x)=\left(\dfrac{\arcsin(\sin^2 x)}{\arccos(\cos^2 x)}\right)^{\operatorname{arctg}^2 x}$ $\qquad f'(x)=?$

2. Нађи тангенту у тачки $x_0=1$ на криву $f(x) = 3x^2+e^x-x$.

$A = (x_0, \overbrace{f(x_0)}^{y_0})$ $\quad f(x_0)=f(1)=3+e^1-1=\boxed{2+e}$



коеф правца $\underline{k=f'(x_0)}$

$$f'(x) = (3x^2+e^x-x)' = 6x+e^x-1$$

$$\boxed{f'(1) = 5+e} \xrightarrow{k}$$

$\left[\ k\text{-коеф пр.} \quad (x_0,y_0)\in p \right.$

$\left. p: \quad y-y_0=k\cdot(x-x_0)\ \right]$

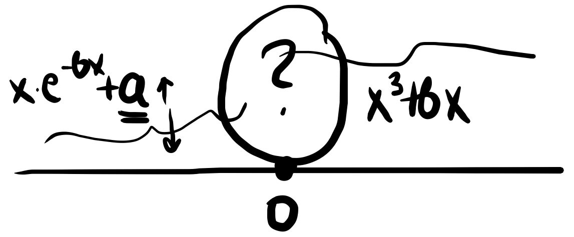
тангента: $\quad y-(2+e)=(5+e)\cdot(x-1)$

$$\boxed{y=(5+e)\cdot x-3}$$

□

③ $a,b\in\mathbb{R}$ $a,b=?$ тд. f буде диф. на $\mathbb{R}$ (ако $\exists$)

$$f(x)=\begin{cases} x\cdot e^{-bx}+a, & x<0\\ x^3+bx, & x\geq 0\end{cases}$$



НЕПРЕКИДНОСТ: $x\neq 0$ ✓

$x=0$: $f(0)=0$

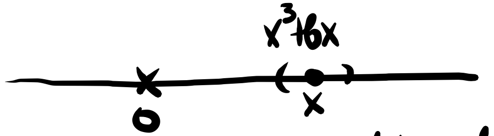
$$\lim_{x\to 0+} f(x)=\lim_{x\to 0+}(x^3+bx)=0$$

$$\lim_{x\to 0-} f(x)=\lim_{x\to 0-}(x\cdot e^{-bx}+a)=a$$

f непр у $x=0$ $\Leftrightarrow$ $\boxed{a=0}$

$\boxed{a=0}$ $\leadsto$ $f(x)=\begin{cases} x\cdot e^{-bx}, & x<0\\ x^3+bx, & x\geq 0\end{cases}$

Диф.: за $x\neq 0$:



$x>0$: $f'(x)=3x^2+b$; $x<0$: $f'(x)=e^{-bx}+x\cdot e^{-bx}(-b)$

$x=0$: $\boxed{f'_-(0)}=\lim_{h\to 0-}\dfrac{f(0+h)-f(0)}{h}=$

$$=\lim_{h\to 0-}\frac{h\cdot e^{-bh}-0}{h}=\lim_{h\to 0-} e^{-bh}=\boxed{1}$$

$$\boxed{f'_+(0)}=\lim_{h\to 0+}\frac{f(h)-f(0)}{h}=\lim_{h\to 0+}\frac{h^3+bh}{h}$$

$$=\lim_{h\to 0+}(h^2+b)=\boxed{b}$$

f диф. у 0 $\Leftrightarrow$ $f'_-(0)=f'_+(0)$ $\Leftrightarrow$ $\boxed{b=1}$

Закључак: f диф. на $\mathbb{R}$ $\Leftrightarrow$ $a=0$ и $b=1$.

④ Испитати диф. f на D_f, $f(x)=\sqrt{1-e^{-x^2}}$

D_f: $1-e^{-x^2}\geq 0 \iff \underset{e^0}{1}\geq e^{-x^2} \iff 0\geq -x^2 \iff \top$

$\boxed{D_f=\mathbb{R}}$

НЕПР. непрекидна на $\mathbb{R}$ ✓

ДИФ.

$$f'(x) = \frac{1}{2}\cdot\frac{1}{\sqrt{1-e^{-x^2}}}\cdot\left(1-e^{-x^2}\right)' = \frac{1}{2}\cdot\frac{1}{\sqrt{1-e^{-x^2}}}\cdot 2xe^{-x^2} = \frac{x e^{-x^2}}{\sqrt{1-e^{-x^2}}}$$

за $1-e^{-x^2}\neq 0$
$\iff 1\neq e^{-x^2}$
$\iff x\neq 0$

за $x\neq 0$ f диф.
"$f'(x)$ $\exists$ x"

остаје: $?\exists f'(0)$

$$f'(0) \underset{\text{ако } \exists}{=} \lim_{h\to 0}\frac{f(h)-f(0)}{h} = \lim_{h\to 0}\frac{\sqrt{1-e^{-h^2}}}{h}$$

$f(0)\to 0$

$\neg\exists$

$\left[\ \frac{e^t-1}{t}\to 1,\ t\to 0\ \right]$

h под корен? $+/-$

$h\to 0_+$: $\boxed{f'_+(0)} = \lim_{h\to 0_+}\frac{\sqrt{1-e^{-h^2}}}{h}$ $\qquad h=\sqrt{h^2}$

$$=\lim_{h\to 0_+}\sqrt{\frac{1-e^{-h^2}}{h^2}} = \lim_{h\to 0_+}\sqrt{\frac{e^{-h^2}-1}{\underbrace{-h^2}_{t}}} \to 1, \quad -h^2\to 0$$

$$=\sqrt{1}=\boxed{1}$$

$h\to 0_-$: $\boxed{f'_-(0)} = \lim_{h\to 0_-}\frac{\sqrt{1-e^{-h^2}}}{h}$ $\qquad h=-\sqrt{h^2}$

$$=\lim_{h\to 0_-} -\sqrt{\frac{1-e^{-h^2}}{h^2}} = \lim_{h\to 0_-} -\sqrt{\frac{e^{-h^2}-1}{-h^2}} \to 1$$

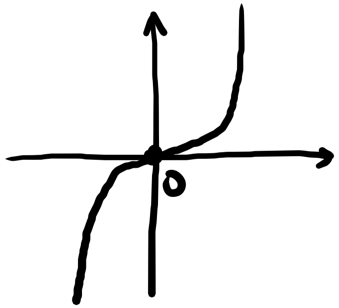
$$=-\sqrt{1}=\boxed{-1}$$

$f'_+(0)\neq f'_-(0) \Rightarrow \neg\exists f'(0)$ и $\boxed{f \text{ није диф. у } 0}$

Закључак: f диф. на $(-\infty,0)\cup(0,+\infty)$ $\square$

5. $f(x)=x\cdot|x|$ диференцијабилност ?

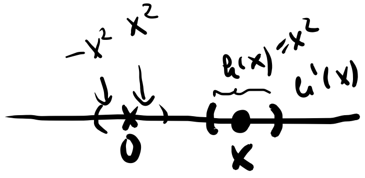
$$f(x)=\begin{cases} x^2, & x\geq 0 \\ -x^2, & x<0 \end{cases}$$



непр: ✓ $\mathbb{R}$

диф: $x\neq 0$:

$$f'(x)=\begin{cases} 2x, & x>0 \\ -2x, & x<0 \end{cases}$$



$x=0$:

$$f'(0) \overset{\text{деф}}{=} \lim_{h\to 0}\frac{f(h)-\overset{=0}{f(0)}}{h}=\lim_{h\to 0}\frac{h\cdot|h|}{h}=\lim_{h\to 0}|h|=0$$

$\exists\, f'(0)=0$

$\Rightarrow$ $\boxed{f \text{ диф. на } \mathbb{R}}$

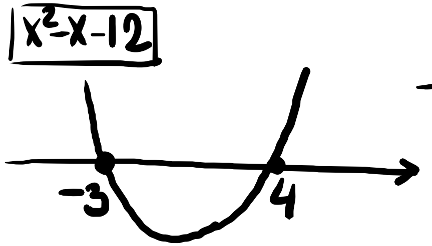
6. диф? $f(x)=|x^2-x-12|\cdot\cos\frac{x\pi}{2}$

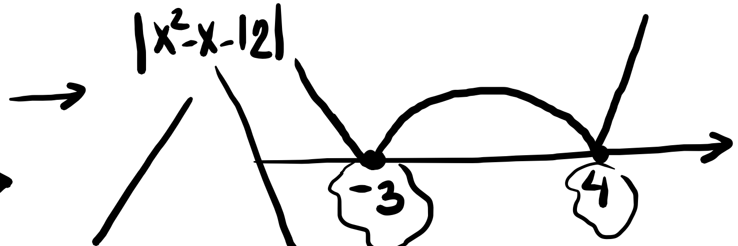
НЕПР: f непр на $\mathbb{R}$ као произв две непр.

Диф.: $x^2-x-12=0$

$(x-4)(x+3)=0$

$x_1=-3 \quad x_2=4$





диф. за $x\neq -3,4$

није диф. за $x_1=-3, x_2=4$

$\Rightarrow \boxed{f \text{ сигурно диф. за } \forall x\neq -3,4}$

$x_1=-3$? $x_2=4$?

$\boxed{x_1=-3}$

$$\lim_{h\to 0}\frac{f(-3+h)-f(-3)}{h} = \lim_{h\to 0}\frac{|(-3+h)^2-(-3+h)-12|\cdot\cos\frac{(-3+h)\pi}{2}}{h}$$

$f(-3)=0$

$\cos\frac{(-3+h)\pi}{2} = -\sin\frac{h\pi}{2}$

$\cos\left(\frac{h\pi}{2}-\frac{3\pi}{2}\right)$

$\cos\left(\alpha-\frac{3\pi}{2}\right)=\cos\left(\alpha+\frac{\pi}{2}\right)$

$=-\sin\alpha$

$$=\lim_{h\to 0}\frac{|h^2-7h|\cdot -\sin\frac{h\pi}{2}}{h} = \lim_{h\to 0}\underbrace{|h^2-7h|}_{\to 0}\cdot\frac{-\sin\frac{h\pi}{2}}{\frac{h\pi}{2}}\cdot\frac{\pi}{2} = 0\cdot(-1)\cdot\frac{\pi}{2}=0$$

$\frac{\sin\frac{h\pi}{2}}{\frac{h\pi}{2}} \to 1$, $\frac{h\pi}{2}\to 0$

$\boxed{f'(-3)=0} \Rightarrow \boxed{f \text{ диф у } x_1=-3}$

$$\boxed{x_2=4}$$

$$\lim_{h\to 0}\frac{f(4+h)-f(4)^{=0}}{h}=\lim_{h\to 0}\frac{|(4+h)^2-(4+h)-12|\cdot\cos\frac{(4+h)\pi}{2}}{h}$$

$$\left[\,f(x)=|x^2-x-12|\cdot\cos\frac{x\pi}{2}\,\right]$$

$$\cos\left(\frac{h\pi}{2}+2\pi\right)=\cos\frac{h\pi}{2}$$

$$=\lim_{h\to 0}\frac{|h^2+7h|\cdot\cos\frac{h\pi}{2}}{h}=\lim_{h\to 0}\frac{|h^2+7h|}{h}$$

$$\cos\frac{h\pi}{2}\to 1$$

$$\left[\,h=\operatorname{sgn}h\cdot|h|\,\right]$$

$$=\lim_{h\to 0}\frac{|h^2+7h|}{\operatorname{sgn}h\cdot|h|}$$

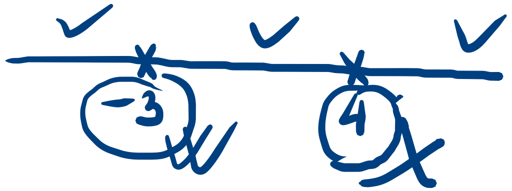
$$=\lim_{h\to 0}\frac{1}{\operatorname{sgn}h}\cdot\underbrace{|h+7|}_{\to 7}$$

$h\to 0+$ $$f'_+(4)=\lim_{h\to 0+}\frac{1}{\underbrace{\operatorname{sgn}h}_{=1}}\cdot|h+7|=7$$

$h\to 0-$ $$f'_-(4)=\lim_{h\to 0-}\frac{1}{\underbrace{\operatorname{sgn}h}_{=-1}}\cdot|h+7|=-7$$

$7 \neq -7$

$\boxed{f \text{ није диф. у } x_2=4}$



f диф. на $(-\infty,4)\cup(4,+\infty)$ ∎

⑦ $f(x)=[x]\cdot\sin\pi x$ непр? диф? на $\mathbb{R}$

НЕПР: $\sin\pi x$ непр на $\mathbb{R}$

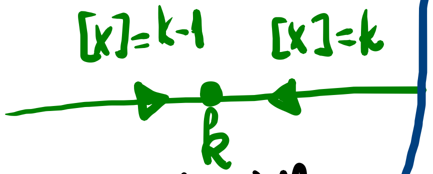
$[x]$ - непр на $\bigcup_{k\in\mathbb{Z}}(k,k+1)$  производ $\Big\}$ f непр. на $\bigcup_{k\in\mathbb{Z}}(k,k+1)$ ✓

$\boxed{x=k\in\mathbb{Z}}$ $f(k)=[k]\cdot\sin k\pi=0$

$$\lim_{x\to k-} f(x)=\lim_{x\to k-}\underbrace{[x]}_{=k-1}\cdot\underbrace{\sin\pi x}_{\to\,\sin\pi k=0}=(k-1)\cdot 0=0 \ \checkmark$$

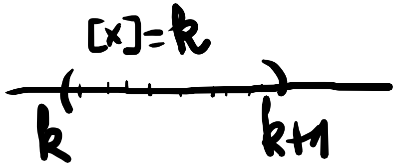
$$\lim_{x\to k+} f(x)=\lim_{x\to k+}\underbrace{[x]}_{k}\underbrace{\sin\pi x}_{\to\,\sin\pi k=0}=0 \ \checkmark$$

$\Rightarrow f$ је непр у k, $\forall k\in\mathbb{Z}$



f непр. на $\mathbb{R}$

ДИФ: $x\notin\mathbb{Z}$ $\quad x\in(k,k+1)$, $k\in\mathbb{Z}$



$\leftarrow \forall x\in(k,k+1)$ $f(x)=k\cdot\sin\pi x$

$f'(x)=k\cdot\cos\pi x\cdot\pi$

$\boxed{f \text{ диф. на } (k,k+1),\ \forall k\in\mathbb{Z}}$

Шта се дешава за $k\in\mathbb{Z}$?

$x = k \in \mathbb{Z}$

$[x]=k-1 \quad [x]=k$

$\lceil f(x) = [x]\cdot \sin \pi x \rfloor$

$$\boxed{f'_+(k)} = \lim_{h\to 0+} \frac{f(k+h) - f(k)}{h} = \lim_{h\to 0+} \frac{[k+h]\sin(k+h)\pi}{h}$$

$\sin(k\pi + h\pi) \qquad \sin(\alpha + k\pi) \; k\in\mathbb{Z}$

$= (-1)^k \sin\alpha$

$$= \lim_{h\to 0+} \frac{k\cdot(-1)^k \cdot \sin h\pi}{h} = \lim_{h\to 0+} k\cdot(-1)^k \cdot \frac{\sin h\pi}{h\pi}\cdot\pi$$

$$= \boxed{k\cdot(-1)^k\cdot\pi}$$

$$\boxed{f'_-(k)} = \lim_{h\to 0-} \frac{f(k+h) - f(k)}{h} = \lim_{h\to 0-} \frac{[k+h]\cdot\sin(k+h)\pi}{h}$$

$[k+h] = k-1$

$\sin(k+h)\pi = (-1)^k \sin h\pi$

$$= \lim_{h\to 0-} \frac{(k-1)\cdot(-1)^k \sin h\pi}{h\pi}\cdot\pi = \boxed{(k-1)\cdot(-1)^k\cdot\pi}$$

$k\in\mathbb{Z}$: $f'_+(k) \neq f'_-(k) \Rightarrow f$ nije diф y k. **Zaključak**: f jeste diф. na $\bigcup_{k\in\mathbb{Z}} (k, k+1)$ $\square$

* za vežbu: $f(x) = \sin x \cdot |\sin x|$ $\quad \leftarrow \sin x = 0$, $x = k\pi$

ИЗВОД ИНВЕРЗНЕ ФУНКЦИЈЕ

T $f:(a,b)\to(\alpha,\beta)$ бијекција, диф. у x_0, $f'(x_0)\neq 0$

$f^{-1}:(\alpha,\beta)\to(a,b)$ диф. у $f(x_0)=y_0$

$$\Rightarrow (f^{-1})'(y_0)=\frac{1}{f'(x_0)}$$

$$x_0 \underset{f^{-1}}{\overset{f}{\rightleftarrows}} y_0=f(x_0)$$

8. $g(y)=\operatorname{arctg} y$, $y\in\mathbb{R}$ $\quad g'(y)=?$

$$\left(-\frac{\pi}{2},\frac{\pi}{2}\right) \underset{g(y)=\operatorname{arctg} y}{\overset{\operatorname{tg} x=f(x)}{\rightleftarrows}} \mathbb{R}$$

$g(y)=f^{-1}(y)$

$f'(x)=(\operatorname{tg}x)'=\dfrac{1}{\cos^2 x}$

$y_0\in\mathbb{R} \to y_0=\operatorname{tg}x_0$, $x_0\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$

$$\underbrace{(\operatorname{arctg})'(y_0)}_{(f^{-1})'(y_0)} \overset{T}{=} \frac{1}{f'(x_0)} = \frac{1}{1/\cos^2 x_0} = \underbrace{\cos^2 x_0}_{\text{изразити преко } y_0=\operatorname{tg}x_0}$$

$$=\frac{1}{1+\operatorname{tg}^2 x_0}=\frac{1}{1+y_0^2}$$

$$\boxed{(\operatorname{arctg})'(y)=\frac{1}{1+y^2}}$$

* $g(y)=\arccos y$, $y\in(-1,1)$
* $g(y)=\arcsin y$, $y\in(-1,1)$

$g'(y)=?$

~ Основне теореме диференцијалног рачуна ~

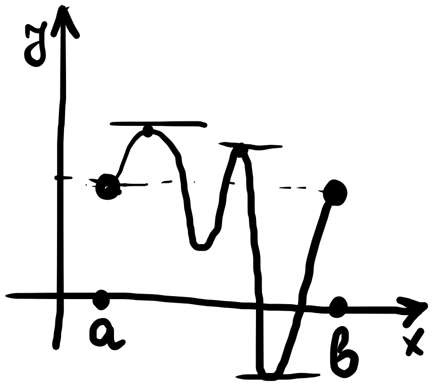
$\boxed{T}$ **Ролова теорема**

$f:[a,b]\to\mathbb{R}$ непрекидна

f диф. на (a,b)

$f(a)=f(b)$

$\Rightarrow \exists c\in(a,b)\ f'(c)=0$



(1) $a_0,a_1,\dots,a_n\in\mathbb{R}$ тд. важи: $a_0+\frac{a_1}{2}+\frac{a_2}{3}+\dots+\frac{a_n}{n+1}=0$

Док. да полином: $p(x)=a_0+a_1x+a_2x^2+\dots+a_nx^n$

има нулу у $\mathbb{R}$.

? $\exists c\in\mathbb{R}\ \ p(c)=0$

($2\nmid n$: p сигурно има реалну нулу)

Правимо ф-ју f тд. $f'(x)=p(x), \forall x$

$a_0=(a_0\cdot x)'$

$a_1 x=\left(a_1\cdot x^2\cdot \frac{1}{2}\right)'$

$\dots\ \ a_k\cdot x^k=\left(a_k\cdot \frac{x^{k+1}}{k+1}\right)'$

$$\boxed{f(x)=a_0x+a_1\cdot\frac{x^2}{2}+a_2\cdot\frac{x^3}{3}+\dots+a_n\cdot\frac{x^{n+1}}{n+1}}$$

$f'(x)=p(x),\ \forall x\in\mathbb{R}$

$$\left.\begin{array}{l} f(0)=0 \\ f(1)=a_0+a_1\cdot\frac{1}{2}+\dots+\frac{a_n}{n+1}=0 \end{array}\right\}\ \boxed{f(0)=f(1)}$$

$[0,1]$

f непр. на $[0,1]$, диф. на $(0,1)$

Рол: $\Rightarrow \exists c\in(0,1)\ f'(c)=0 \Rightarrow \boxed{p(c)=0}$